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6. INTERESTING PROPERTIES

The class of automatic groups encompasses a wide variety of groups, many arising from vastly different geometric situations and exhibiting vastly different behaviors. It is indeed surprising that such a diverse class of groups should share all of the properties which come from being automatic.

SOME NICE PROPERTIES OF AUTOMATIC GROUPS:

1. *Finite presentation.* Although it is true *a priori* only that automatic groups are finitely generated, it is in fact true that every automatic group has a finite presentation; that is, a presentation with finitely many generators and finitely many relators.

2. *Fast solution to word problem.* There is a quadratic time algorithm to solve the word problem for an automatic group; quadratic in the sense that one can check in $O(n^2)$ steps whether a word of length n represents the identity or not. In fact, one can put a word in a canonical form (given by the regular language of the word acceptor) in quadratic time ([E *et al.*]). We should also mention that, if G is automatic, then there is some automatic structure for which there is a unique canonical form for each group element; that is, the natural map $\pi: L \to G$ is bijective. It is not known if automatic groups have solvable conjugacy problem.

3. *Fast pictures.* One can enumerate the elements of an automatic group with a unique word for each element; the first n elements can be enumerated in time $O(n \log n)$. This allows for efficient construction of the Cayley graph of an automatic group, as well as pictures of sets which are invariant under the action of an automatic group. Mumford and Wright have used automatic groups in programs to draw pictures of limit sets of quasifuchsian groups efficiently; mathematicians have been trying to draw such pictures efficiently on computers for many years. Automatic group structures have also been used in relating frames in the soon to be released movie "Not Knot". After implementing ideas from automatic groups, the time for computing the numbers used to locate points in a single frame went from about 20 minutes to about 15 seconds on a four processor Iris. Since there are 28 frames per second, the amount of time saved is quite substantial ([Ep2]).

4. *Uniform algorithms.* There is an algorithm which takes as input a finite presentation and as output gives the automatic structure $(W, W_=, W_{a_1}, \ldots, W_{a_n})$ for the group. The algorithm does not terminate if the group is not automatic; in fact, there is no algorithm which can determine

for a presentation $G = < x_1, \ldots, x_n: r_1, \ldots, r_m >$ of a group G whether or not G is automatic. The algorithm exists since it is possible to give a list of (checkable) axioms characterizing when a collection of finite state automata form the automatic structure of a group. The algorithm for solving the word problem for a fixed automatic group is itself algorithmically constructible, so there is a uniform algorithm for solving the word problem for all automatic groups! It should be noted that the algorithm which finds the automatic structure from the presentation is extremely slow. More efficient methods for finding automatic structures in slightly more special cases have in fact been programmed by Epstein, Holt and Rees. Their ideas involve the Knuth-Bendix process, and their programs have been quite successful at finding automatic structures for a number of examples ([EHR]). One can also show that there is an algorithm which takes as input a presentation of an (*a priori*) automatic group, and as output tells whether or not the group is trivial, and whether or not the group is finite. For presentations in the class of arbitrary groups, these problems have been shown to be unsolvable ([Ra]). It is an open question whether the isomorphism problem is solvable for automatic groups; that is, if there is an algorithm to determine, given two presentations whose groups are automatic, whether or not the groups are isomorphic.

5. *Rational growth functions.* Many automatic groups have rational growth functions; in particular those groups which have an automatic structure where the language of accepted words consists of geodesics have rational growth functions. Recall that if we are given a presentation (G, S), and if c_n denotes the number of elements of $\Gamma_S(G)$ at distance n from the identity, the counting function for (G, S) is the function $f(x) = \sum_{i=1}^{\infty} c_n x^n$. The rationality of $f(x)$ may be interpreted as the fact that the number of elements contained in the sphere (or ball) of radius n in $\Gamma_S(G)$ may be determined by a linear recursion, such as the recursion which gives the Fibonacci sequence. The automatic groups team at Warwick has programs which, given the automatic structure of an automatic group where the language of accepted words consists of geodesics, produces a rational function giving the growth of the group. It is an open question whether all automatic groups have rational growth functions.

6. *Type FP_∞*. If G is an automatic group, then there exists an Eilenberg-MacLane space $K(G, 1)$ with finitely many cells in each dimension; in this case G is said to be of type FP_∞ (see [Al]). It still seems to be unknown whether torsion-free automatic groups must have finite cohomological dimension.

7. *Quadratic isoperimetric function.* We shall not discuss isoperimetric functions in groups here; the reader may consult [Ge1, Sh, E *et al.*]. Isoperimetric functions in groups are extremeley interesting, and have become quite important in combinatorial group theory and geometry; related concepts have recently proven useful in the study of three-manifolds ([Ge2, St]). Gromov showed that the negatively curved groups are precisely those which have a linear isoperimetric function. Automatic groups satisfy a quadratic isoperimetric function, but are not characterized by this property. Thurston (unpublished) has shown that the five-dimensional Heisenberg group has a quadratic isoperimetric function but is not automatic.

It is possible to get a feel for the breadth and unifying power of the theory of automatic groups by matching up groups in the list of examples with properties from the list just given. For instance, automatic groups give a uniform quadratic time solution to the word problem for fundamental groups of compact negatively curved manifolds, most Coxeter groups, and the braid groups (previously known algorithms for the braid groups never discussed speed, and seem to be much slower). The reader may wish to contemplate other "theorems" obtained by matching pairs in the lists.

7. Related topics, open problems, and a vision of the future

The field of automatic groups (and related topics) is still quite young; accordingly, there are many open questions which are interesting, easy to state, and perhaps not so difficult for a newcomer to think about. Listed below are a few personal favorites. For other open questions, the reader is encouraged to dive into the references given at the end of this paper, in particular [Ge3].

Some open problems:

1. Prove that the mapping class groups of hyperbolic surfaces are automatic. As a (perhaps) easier question, show that these groups satisfy a quadratic isoperimetric inequality ([Ge1, E *et al.*]).

2. Are cocompact lattices in $SL_3(\mathbf{R})$ automatic? Note that $SL_3(\mathbf{Z})$ is a lattice in $SL_3(\mathbf{R})$ which is not cocompact and not automatic. There is a p-adic analog to this question which has been solved ([GS1]). Find examples of other arithmetic groups which are or are not automatic. So far not much seems to be known for arithmetic groups, except for a result of Gersten and Short ([GS3]) which shows that $SL_2(\mathscr{O})$, with $\mathscr{O}$ a real quadratic number field, is