# §1. Category

THEOREM 1. *If* $M$ *is a (compact) nilmanifold, then* $\text{cat}(M) = \dim(M) = \text{rank}(\pi_1 M)$.

Hence, the best possible result which Lusternik-Schnirelmann theory can provide for nilmanifolds is the immediate.

COROLLARY. *The number of critical points of a smooth function on a (compact) nilmanifold* $M$ *is bounded below by* $\text{rank}(\pi_1 M) + 1$.

In fact, Theorem 1 was announced for all $K(\pi, 1)$'s by Eilenberg and Ganea [11]. Unfortunately, details of the proofs of their three fundamental propositions never appeared, thus contributing, I believe, to the ignorance of the result among the dynamicists and topologists of today. Indeed, this paper was originally written in response to Chris McCord's question and without knowledge of the Eilenberg-Ganea result. Furthermore, in looking at the Eilenberg-Ganea propositions, it is difficult to see the relationship between the structures of $\pi$ and $K(\pi, 1)$ and the consequent determination of category as $\text{rank}(\pi)$. I hope that the approach of this paper will remedy this defect, at least in the case of nilmanifolds. The beautiful structure theory of nilmanifolds (i.e. finitely generated torsionfree nilpotent groups) is ideally suited for an approach in terms of minimal models. In fact, in some sense, this paper is simply an exposition of just how well rational homotopy theory and nilmanifold theory fit together (in the representative situation of determining category).

Theorem 1 will be given a simple ("up to" the machinery of rational homotopy theory) proof in §4. Since this paper is written for workers in dynamical systems, I have tried to make it somewhat self-contained. Therefore, §1 and §2 are devoted to recollections on category and its rational homotopy description respectively. §3 recollects structural knowledge of nilmanifolds and §5 presents an analogue of Theorem 1 for iterated principal bundles. (The basic reference for the rational homotopy version of L.S. category is [3]; I have attempted to cull the essential ingredients for the proof of Theorem 1, but the reader will find other interesting applications in that work. Also see [2].)

## §1. CATEGORY

The *category* of a space $M$, $\text{cat}(M)$, is the least integer $m$ so that $M$ is covered by $m + 1$ open subsets each of which is contractible within $M$.

An equivalent definition (at least for the spaces we consider here) was given by G. Whitehead (see [10]): Let $M^{m+1}$ denote the $(m + 1)$-fold product and let $T^{m+1}(M)$ denote the subspace consisting of all $(m+1)$-tuples $(x_1, \cdots, x_{m+1})$ with at least one $x_i$ equal to a specified basepoint in $M$. ($T^{m+1}(M)$ is usually called the "fat wedge".) In particular, $T^2(M) = M \vee M$; two copies of $M$ attached at the specified basepoint. Now let $\triangle : M \to M^{m+1}$ denote the $(m+1)$-fold diagonal $\triangle(x) = (x, x, \cdots, x)$ and $j: T^{m+1}(M) \to M^{m+1}$ the natural inclusion. Whitehead's definition is then: $\mathrm{cat}(M)$ is the least integer $m$ so that, up to homotopy, $\triangle$ factors through the fat wedge; that is, there exists $\triangle' : M \to T^{m+1}(M)$ with $j\triangle' \simeq \triangle$.

The *cuplength* of $M$, $\mathrm{cup}(M)$, is the largest integer $k$ so that there exist $x_i \in H^{n_i}(M; R)$, $i = 1, \cdots, k$ and a nontrivial cup-product

$$0 \neq x_1 x_2 \cdots x_k .$$

The following result is well-known and is the basis of many calculations of category:

PROPOSITION. $\mathrm{cup}(M) \leq \mathrm{cat}(M)$.

For a proof, see [10] for example. Other important properties of category are:

(1) Category is an invariant of homotopy type.

(2) If $C_f = Y \cup_f CX$ is a mapping cone, then $\mathrm{cat}(C_f) \leq \mathrm{cat}(Y) + 1$.

(3) If $X$ is a $CW$-complex, then (by induction on skeleta and (2)) $\mathrm{cat}(X) \leq \dim X$.

(4) In fact, (3) may be generalized: If $X$ is $(r-1)$-connected, then $\mathrm{cat}(X) \leqslant (\dim X)/r$.

The proofs of these properties are straightforward; see [10] for example. In particular, we shall use (3) in our determination of the category of nilmanifolds.

*Examples*

1. $\mathrm{cat}(X) = 0$ if and only if $X$ is contractible.
2. $\mathrm{cat}(S^n) = 1$.
3. More generally, $\mathrm{cat}(X) = 1$ if and only if $X$ is a nontrivial co-$H$ space.
4. $\mathrm{cat}(T^n) = n$ (this follows from the proposition and property (3) above).

We single out an example of interest in dynamical systems which, although quite simple, does not seem to be well known among dynamicists. (The analogue for Kähler manifolds *is* well known among topologists.)

5. If $M^{2n}$ is a simply connected compact symplectic manifold, then $\operatorname{cat}(M) = n = \frac{1}{2}\dim(M)$. (First, observe that the volume form is not exact since it represents a nontrivial fundamental class of $M$. Because $\omega^n/n! = \text{vol}$ (see [1], p. 165), the nondegenerate closed 2-form $\omega$ cannot be exact either. Hence, $\omega^n$ represents a nontrivial cup-product of length $n$ in $\mathbf{R}$-cohomology. By property (4) above, $\operatorname{cat}(M) \leqslant (\dim M)/2 = n$. Hence,

$$n \leq \operatorname{cup}(M) \leq \operatorname{cat}(M) \leq \frac{1}{2}\dim M = n ,$$

and the result follows.)

## §2. Rational homotopy and category

The basic reference for this section is [3]. To each space $X$, Sullivan functorially associated a *c*ommutative *d*ifferential *g*raded *a*lgebra $(A(X), d)$ of rational polynomial forms possessing the salient property that integration defines a natural algebra isomorphism between $H^*(A(X), d)$ and $H^*(X; \mathbf{Q})$. Furthermore, the cdga $A(X)$ was shown to contain all the rational homotopy information about $X$; information which may be gleaned from an associated cdga *minimal model* of $A(X)$.

A cdga $(\Lambda, d)$ is *minimal* if (1) $\Lambda = \Lambda X$, where $X = \oplus_{i>0} X^i$ is a graded $\mathbf{Q}$-vector space and $\Lambda X$ denotes that $\Lambda$ is freely generated by $X$; that is, $\Lambda X$ = Symmetric algebra $(X^{\text{even}}) \otimes$ Exterior algebra $(X^{\text{odd}})$. (2) There is a basis for $X$, $\{x_\alpha\}_{\alpha \in I}$, so that if $I$ is well ordered by $<$, then $dx_\beta \in \Lambda^+_{\alpha<\beta}(x_\alpha) \cdot \Lambda^+_{\alpha<\beta}(x_\alpha)$. That is, $\Lambda$ is constructed by stages and the differentials of $\beta^{\text{th}}$ stage generators are decomposable in the generators of previous stages.

A *minimal model* for a space $M$ is a minimal cdga $\Lambda(M)$ and a cdga map $\Lambda(M) \to A(M)$ inducing an isomorphism in cohomology. The fundamental theorem of rational homotopy theory is then (see [4] for example).

THEOREM. *Each space $M$ has a minimal model $\Lambda(M)$ and, furthermore, for nilpotent spaces the stage by stage construction precisely mirrors the rational Postnikov tower with the differential corresponding to the k-invariant.*

Recall that a space $M$ is *nilpotent* if its fundamental group $\pi_1(M)$ is a nilpotent group and the natural action of $\pi_1(M)$ on $\pi_n(M)$ (see [10]) is a nilpotent action (see [12]). In particular, any simply connected space or any $K(\pi, 1)$ with $\pi$ nilpotent is a nilpotent space. The theorem then says that, for