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AND PROGRESSION-FREE SEQUENCES OF INTEGERS
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Example. Let B be the sequence $(0\omega 1\omega)^\infty$, and let f be defined on $\{0, 1, \omega\}$ by $f(0) = 1$, $f(1) = 0$, $f(\omega) = \omega$, then one has:

$$B_0 = (0\omega 1\omega 0\omega 1\omega \cdots) (= B),$$

$$B_1 = (1\omega 0\omega 1\omega 0\omega \cdots),$$

$$B_2 = B_0,$$

...

$$A_0 = (0\omega 1\omega 0\omega 1\omega \cdots),$$

$$A_1 = (011\omega 001\omega \cdots),$$

$$A_2 = (0110001\omega \cdots),$$

...

Note that if Γ is finite, and f one-to-one, such a sequence $Tt(B, f)$ can also be obtained by replacing B by a sequence of greater period and f by id .

We now give four examples of Toeplitz transforms in (apparently) unrelated domains.

2. PAPERFOLDING SEQUENCES AND TOEPLITZ TRANSFORMS

In [23] and [22] Prodinger and Urbanek study the Toeplitz transform of $((0\omega 1\omega)^\infty, id)$ and of $((0\omega 1\omega 1\omega 0\omega)^\infty, id)$. They prove that these sequences do not have arbitrarily long squares (a sequence A contains a square of length $2k$ if there exists an index j such that $A(j+n) = A(j+n+k)$ for every n between 0 and $k-1$). Dekking already noticed in [10] that the first sequence is nothing but the regular paperfolding sequence (see [9], [18], [20], [17]), which is obtained by repeatedly folding a piece of paper, and we obtained in [1] the same result as Prodinger and Urbanek for the general paperfolding sequences. Let us give here two simple examples:

PROPOSITION. *Let B be the sequence $B = (0\omega 1\omega)^\infty$ and let f be defined by $f(0) = 1$, $f(1) = 0$ and $f(\omega) = \omega$. Then*

the sequence $Tt(B, id)$ is the regular paperfolding sequence,

the sequence $Tt(B, f)$ is the alternate paperfolding sequence.

Proof. It follows from instance from [18] (after replacing 1's by 0's and - 1's by 1's) that the regular paperfolding sequence R and the alternate paperfolding sequence A are given by

$$R(2^k(2m+1)-1) = \frac{(1 - (-1)^m)}{2} \quad \forall k, m \geq 0.$$

$$A(2^k(2m+1)-1) = \frac{(1 - (-1)^{k+m})}{2} \quad \forall k, m \geq 0.$$

Let U and V be the sequences defined by

$$U = Tt((0\omega 1\omega)^\infty, id),$$

$$V = Tt((0\omega 1\omega)^\infty, f).$$

A straightforward computation gives

$$U(2n) = \frac{(1 - (-1)^n)}{2} \quad \forall n \geq 0,$$

$$U(2n+1) = U(n) \quad \forall n \geq 0.$$

Hence

$$\begin{aligned} U(2^k(2m+1)-1) &= U(2(2^{k-1}(2m+1)-1)+1) \\ &= U(2^{k-1}(2m+1)-1) = \dots = U(2m) = \frac{(1 - (-1)^m)}{2}. \end{aligned}$$

This proves that $U = R$.

In the same way one has

$$V(2n) = \frac{(1 - (-1)^n)}{2},$$

$$V(2n+1) = 1 - V(n).$$

Hence

$$\begin{aligned} V(2^k(2m+1)-1) &= V(2(2^{k-1}(2m+1)-1)+1) = 1 - V(2^{k-1}(2m+1)-1) \\ &= \dots = \left\{ \begin{array}{ll} V(2m) & \text{if } k \text{ is even,} \\ 1 - V(2m) & \text{if } k \text{ is odd,} \end{array} \right\} = \frac{(1 - (-1)^{k+m})}{2}, \end{aligned}$$

and finally $V = A$.