3. Iteration of continuous functions and Toeplitz transforms

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3. Iteration of continuous functions and Toeplitz transforms

When iterating a unimodal (i.e. increasing then decreasing) and continuous map of the interval $[0, 1]$, say F_μ, depending on the parameter μ, one knows that, assuming certain properties of the map $\mu \to F_\mu$, a Feigenbaum doubling cascade phenomenon occurs (see [7] for instance): when the parameter increases, the function has first an attractive fixed point for $\mu_0 \leqslant \mu < \mu_1$, then an attractive cycle of length 2 for $\mu_1 \leqslant \mu < \mu_2$, then an attractive cycle of length 4, ... There is a "first" value of the parameter μ_∞ for which a "chaotic" behaviour appears, and this value is the limit of the sequence $(\mu_n)_n$. This sequence grows roughly like a constant term plus a geometric progression C^n, where the constant C is universal, provided that the functions F_μ are smooth enough. This constant is called the Feigenbaum constant.

The orbit of the point 1 under F_{μ_∞} can be coded by a universal binary sequence A (even in cases where the Feigenbaum constant does not appear). This sequence $A = (A(n))$ is defined as 0 if $F_{\mu_\infty}^{(n)}(1)$ is smaller than the point where F_{μ_∞} takes its maximum, and 1 if $F_{\mu_\infty}^{(n)}(1)$ is larger than this value, and does not depend on the family of functions (F_μ). Moreover it has been noticed in [3] that the sequence A is related to the Prouhet-Thue-Morse sequence C (see [6] and its bibliography) by

$$C(n+1) = \sum_{0 \leqslant j \leqslant n} A(j) \text{ modulo } 2 .$$

(Let us recall that C is the fixed point beginning by 0 of the 2-substitution $0 \to 01, 1 \to 10$.)

Actually as noticed in [24], A is the fixed point of the 2-substitution

$$1 \to 10, \quad 0 \to 11 .$$

PROPOSITION. *Let f be defined by $f(0) = 1, f(1) = 0, f(\omega) = \omega$, then the sequence A is the Toeplitz transform of $((1\omega)^\infty, f)$.*

Proof. As the fixed point of the 2-substitution $1 \to 10$, $0 \to 11$, the sequence A can be recursively defined by

$$A(2n) = 1 , \quad A(2n+1) = 1 - A(n) .$$

Remark. The relation between C and A can also be written

$$A(n) = C(n) + C(n+1) \text{ modulo } 2 .$$

If, instead of C one takes a "generalized" Morse sequence C', and if one defines

$$A'(n) = C'(n) + C'(n+1) \text{ modulo } 2 \,,$$

then A' is also a Toeplitz sequence, as proved in [16].

4. Towers of Hanoi and Toeplitz sequences

The tower of Hanoi puzzle consists of three vertical pegs and of N circular disks of different diameters stacked in decreasing order on the first peg. At each step one may transfer the topmost disk from a peg to a different peg according to the rule: no disk is allowed to be on a smaller one. The game ends when all the disks are stacked on the second or third peg.

The sequence of moves for the classical (minimal) Hanoi tower algorithm can be generated in a very easy way as it is 2-automatic (see [4] and section 6), which essentially means that the k^{th} move can be predicted by a machine with bounded memory. More precisely number the pegs as I, II, III and define a (respectively b, c) to be the move which takes the topmost disk from peg I (respectively II, III) and puts it on peg II (respectively III, I). Let $\bar{a}, \bar{b}, \bar{c}$ be the respective opposite moves. Then the sequence of moves for N disks is the prefix of length $2^N - 1$ of an infinite sequence U which is 2-automatic. Moreover the following proposition is proved in [4]:

PROPOSITION. *The infinite sequence of moves U is equal to the Toeplitz transform of* $((a\bar{c}b\omega\, c\bar{b}a\omega\, b\bar{a}c\omega)^{\infty}, id)$.

Note that, keeping the notations of [4], the sequence U is indexed by $1, 2, \cdots$ and not by $0, 1, 2, \cdots$ as the sequences above.

5. Progression-free sequences and Toeplitz sequences

The question of finding a sequence of integers without arithmetic progressions of given length has been intensively studied (see its history in [14] and the included bibliography). In particular what is the "minimal" increasing sequence having this property?

One knows that, if k is a prime number, the minimal sequence of integers without any arithmetic progression of k terms is exactly the increasing sequence of the integers without the digit $k-1$ in their base k expansion (cited