§2. Discontinuous invariants

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$$Y \times Z \to \Omega X \times \Omega X \xrightarrow{*} \Omega X .$$

Then this represents $u * v \in H_{p+q}(\Omega X; \mathbf{Z})$. On the other hand, the composition

$$Y \times Z \to \Omega X \times \Omega X \to PX \times \Omega X \to X \times \Omega X$$

bounds $SY \times Z \to PX \times \Omega X \to X \times \Omega X$, which represents $s(u) \otimes v$. Hence $s(u) \otimes v$ and $u * v$ are related under ∂^{p+1}.

§2. Discontinuous invariants

First we review the definition by Morita ([10]) of discontinuous invariants arising from the Godbillon-Vey invariant for codimension one foliations.

Let $\mathscr{F}$ be a codimension one foliation of a closed oriented $3k$-dimensional manifold M. Then the Godbillon-Vey class $gv(\mathscr{F}) \in H^3(M; \mathbf{R})$ is defined ([6]). Let $\{x_1, \ldots, x_n\}$ be a basis of $H^3(M; \mathbf{Q})$. Then $gv(\mathscr{F})$ is written as

$$gv(\mathscr{F}) = a_1 x_1 + \ldots + a_n x_n ,$$

where $a_1, \ldots, a_n \in \mathbf{R}$. The discontinuous invariant GV_k is defined by

$$GV_k(\mathscr{F}) = \sum_{i_1 < \ldots < i_k} (x_{i_1} \cup \ldots \cup x_{i_k})[M]\; a_{i_1} \wedge_{\mathbf{Q}} \ldots \wedge_{\mathbf{Q}} a_{i_k} \in \mathbf{R}^{\wedge k} = \overbrace{\mathbf{R} \wedge_{\mathbf{Q}} \ldots \wedge_{\mathbf{Q}} \mathbf{R}}^{k} ,$$

where $[M] \in H_{3k}(M; \mathbf{Z})$ is the fundamental class. Morita showed that GV_k is natural, GV_k depends only on the foliated cobordism class of $\mathscr{F}$, and hence there is a universal map $GV_k \colon H_{3k}(B\Gamma_1; \mathbf{Z}) \to \mathbf{R}^{\wedge k}$ ([10]).

The same argument applies to transversely piecewise linear foliations and the discrete Godbillon-Vey class defined in [5] and [3]. Then the following theorem is obtained from the description by Greenberg ([7]) of the classifying space for them and Lemma (1.1).

Theorem (2.1). *Let $\mathscr{F}$ be a codimension one transversely orientable transversely piecewise linear foliation of a closed oriented $3k$-dimensional manifold $M (k \geqslant 2)$. Then $GV_k(\mathscr{F}) = 0$.*

Proof. The weak homotopy type of the classifying space $B\bar{\Gamma}_1^{PL}$ for codimension one transversely oriented transversely piecewise linear foliations is known by Greenberg ([7]). This classifying space $B\bar{\Gamma}_1^{PL}$ has the weak homotopy type of the join $B\mathbf{R}^\delta * B\mathbf{R}^\delta$ of two copies of $B\mathbf{R}^\delta = K(\mathbf{R}, 1)$. Let

gv denote the discrete Godbillon-Vey class defined as a 3-dimensional cohomology class of this classifying space ([5], [3]).

$$gv \in H^3(B\bar{\Gamma}_1^{PL}; \mathbf{R}) .$$

By Lemma (1.1), the higher discontinuous invariants GV_k are trivial in this classifying space $B\bar{\Gamma}_1^{PL}$. Hence by the naturality of GV_k, $GV_k(\mathcal{F}) = 0$.

COROLLARY (2.2). *Let* $\mathcal{F}$ *be a codimension one transversely piecewise linear foliation of* $S^3 \times S^3$. $GV(\mathcal{F}) = (a, b) \in H^3(S^3 \times S^3, \mathbf{R})$ *satisfies* $a/b \in \mathbf{Q} \cup \{\infty\}$.

Proof. $0 = GV_2(\mathcal{F}) = a \wedge_{\mathbf{Q}} b$. Hence $a/b \in \mathbf{Q} \cup \{\infty\}$.

Remark. Morita translated the question of rationality into that of graded commutativity of $*$-product defined on the homology of the group of diffeomorphisms of $\mathbf{R}$ with compact support ([10]). In the later sections, we calculate the homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support as well as the $*$-product structure. We see that the $*$-product is certainly not graded commutative, which insures the rationality. The argument on the rationality of transversely piecewise linear foliations uses the fact that the Godbillon-Vey invariant localizes on transversely discrete sets and this argument cannot be generalized for smooth foliations for the moment. See how the class $C^4_{(1,1,1,1)}$ exists in §3. We also see that the Whitehead product of elements of $\pi_n(B\bar{\Gamma}_1^{PL})$ which are not zero in homology is usually nontrivial and has infinite order.

Remark. The Hurewicz map

$$\pi_n(B\bar{\Gamma}_1^{PL}) \to H_n(B\bar{\Gamma}_1^{PL}; \mathbf{Z})$$

is surjective. To see this, note first that by Greenberg ([7]),

$$H_n(B\bar{\Gamma}_1^{PL}; \mathbf{Z}) \cong \sum_{i=1}^{n-1} \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge n-1-i} .$$

An element $(a_1 \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} a_i) \otimes_{\mathbf{Q}} (b_{i+1} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} b_{n-1}) \in \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge n-1-i}$ is represented by the following foliation of $T^i * T^{n-1-i}$. Consider the foliated $\mathbf{R}$-product with noncompact support over T^{n-1} such that the holonomy $h: \pi_1(T^{n-1}) \to PL(\mathbf{R})$ is given by

$$h(e_j)(x) = e^{a_j}x \text{ for } x < 0 \text{ and } h(e_j)(x) = x \text{ for } x > 0 \text{ if } j = 1, \dots, i$$
$$h(e_j)(x) = x \text{ for } x < 0 \text{ and } h(e_j)(x) = e^{b_j}x \text{ for } x > 0 \text{ if } j = i+1, \dots, n-1 .$$

This foliation restricted to $T^{n-1} \times [-1, 1]$ induces a foliation of $T^i * T^{n-1-i}$ which is

$$T^{n-1} \times [-1, 1]/(T^i \times T^{n-1-i} \times \{-1\} \sim T^i \times \{-1\},\ T^i \times T^{n-1-i} \times \{1\} \sim T^{n-1-i} \times \{1\}) .$$

Note that there is a degree one map from the suspension of T^{n-1} to $T^i * T^{n-1-i}$. Since we can embed $T^{n-1} \times [-1, 1]$ in S^n, we have a degree one map from S^n to the suspension of T^{n-1}, hence to $T^i * T^{n-1-i}$. Thus Hurewicz map is surjective.

§3. Homology of the group of piecewise linear homeomorphisms

Let $PL_c(\mathbf{R})$ denote the group of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support. Let $\mu : PL_c(\mathbf{R}) \times PL_c(\mathbf{R}) \to PL_c(\mathbf{R})$ be the composition of two isomorphisms $PL_c(\mathbf{R}) \cong PL_c((-\infty, 0))$ and $PL_c(\mathbf{R}) \cong PL_c((0, \infty))$, and the inclusion

$$PL_c((-\infty, 0)) \times PL_c((0, \infty)) \to PL_c(\mathbf{R}) .$$

Then μ induces a product $*$ on the homology of $BPL_c(\mathbf{R})^\delta$ ([10]).

The homology of the group $PL_c(\mathbf{R})$ of piecewise linear homeomorphisms of $\mathbf{R}$ with compact support is described as follows. For positive integers i and j, put

$$V^{i,j} = \mathbf{R}^{\wedge i} \otimes_{\mathbf{Q}} \mathbf{R}^{\wedge j} = \overbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}^{i} \otimes_{\mathbf{Q}} \overbrace{(\mathbf{R} \wedge_{\mathbf{Q}} \dots \wedge_{\mathbf{Q}} \mathbf{R})}^{j} .$$

THEOREM (3.1).

$$H_m(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \cong \sum V^{k_1^-, k_1^+} \otimes_{\mathbf{Q}} \dots \otimes_{\mathbf{Q}} V^{k_s^-, k_s^+} ,$$

where the sum is taken over even number of positive integers

$$(k_1^-, k_1^+, \dots, k_s^-, k_s^+)$$

such that $k_1^- + k_1^+ + \dots + k_s^- + k_s^+ = m$. *Moreover, the $*$-product*

$$* : H_i(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \times H_j(BPL_c(\mathbf{R})^\delta; \mathbf{Z}) \to H_{i+j}(BPL_c(\mathbf{R})^\delta; \mathbf{Z})$$

coincides with the tensor product.