§2. Rational homotopy and category

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5. If M^{2n} is a simply connected compact symplectic manifold, then $\mathrm{cat}(M) = n = \frac{1}{2}\dim(M)$. (First, observe that the volume form is not exact since it represents a nontrivial fundamental class of M. Because $\omega^n/n! = \mathrm{vol}$ (see [1], p. 165), the nondegenerate closed 2-form ω cannot be exact either. Hence, ω^n represents a nontrivial cup-product of length n in $\mathbf{R}$-cohomology. By property (4) above, $\mathrm{cat}(M) \leqslant (\dim M)/2 = n$. Hence,

$$n \leq \mathrm{cup}(M) \leq \mathrm{cat}(M) \leq \frac{1}{2}\dim M = n\ ,$$

and the result follows.)

§2. Rational homotopy and category

The basic reference for this section is [3]. To each space X, Sullivan functorially associated a *c*ommutative *d*ifferential *g*raded *a*lgebra $(A(X), d)$ of rational polynomial forms possessing the salient property that integration defines a natural algebra isomorphism between $H^*(A(X), d)$ and $H^*(X; \mathbf{Q})$. Furthermore, the cdga $A(X)$ was shown to contain all the rational homotopy information about X; information which may be gleaned from an associated cdga *minimal model* of $A(X)$.

A cdga (Λ, d) is *minimal* if (1) $\Lambda = \Lambda X$, where $X = \oplus_{i>0} X^i$ is a graded $\mathbf{Q}$-vector space and ΛX denotes that Λ is freely generated by X; that is, $\Lambda X =$ Symmetric algebra $(X^{\mathrm{even}}) \otimes$ Exterior algebra (X^{odd}). (2) There is a basis for X, $\{x_\alpha\}_{\alpha \in I}$, so that if I is well ordered by $<$, then $dx_\beta \in \Lambda^+_{\alpha<\beta}(x_\alpha) \cdot \Lambda^+_{\alpha<\beta}(x_\alpha)$. That is, Λ is constructed by stages and the differentials of β^{th} stage generators are decomposable in the generators of previous stages.

A *minimal model* for a space M is a minimal cdga $\Lambda(M)$ and a cdga map $\Lambda(M) \to A(M)$ inducing an isomorphism in cohomology. The fundamental theorem of rational homotopy theory is then (see [4] for example).

THEOREM. *Each space M has a minimal model $\Lambda(M)$ and, furthermore, for nilpotent spaces the stage by stage construction precisely mirrors the rational Postnikov tower with the differential corresponding to the k-invariant.*

Recall that a space M is *nilpotent* if its fundamental group $\pi_1(M)$ is a nilpotent group and the natural action of $\pi_1(M)$ on $\pi_n(M)$ (see [10]) is a nilpotent action (see [12]). In particular, any simply connected space or any $K(\pi, 1)$ with π nilpotent is a nilpotent space. The theorem then says that, for a nilpotent space, the minimal model is a perfect reflection of the rational homotopy type of the space (eg for $i > 1$, $X^i \cong \mathrm{Hom}(\pi_i(M), \mathbf{Q})$, where $\pi_i(M)$ is the i^{th} homotopy group of M). The minimal model $\Lambda(M)$ is therefore an algebraic version of the $\mathbf{Q}$-localization of M. Indeed, a notion of cdga homotopy may be described so that there is a categorical equivalence between the homotopy categories of rational nilpotent spaces and minimal cdga's.

Examples. (1) $\Lambda(S^{2n+1}) = \Lambda(x_{2n+1}), dx = 0.$

(2) $\Lambda(S^{2n}) = \Lambda(x_{2n}, y_{4n-1}), dy = x^2.$

(3) $\Lambda(\mathbf{C}P(n)) = \Lambda(x_2, y_{2n+1}), dy = x^{n+1}.$

(4) $\Lambda(T^n) = \Lambda(x_1^1, x_1^2, \cdots, x_1^n), d = 0.$

In the next setion we will describe the minimal model of a nilmanifold in terms of the structure of its defining nilpotent group.

In order to understand category in the framework of minimal models, assume for the moment that $\mathrm{cat}(M) = m$. The Whitehead diagram

$$(*)\qquad \begin{array}{ccc} M & \xrightarrow{\Delta} & M^{m+1} \\ & \searrow^{\Delta'} & \uparrow j \\ & & T^{m+1}(M) \end{array}$$

translates (via Sullivan's categorical equivalence) into a homotopy commutative diagram of minimal cdga's,

$$(**)\qquad \begin{array}{ccc} \Lambda X & \leftarrow & (\Lambda X)^{\otimes m+1} \\ & \nwarrow_{\rho} & \downarrow \xi \\ & & \Lambda Y \end{array}$$

where $\Lambda(M) = \Lambda X$, $\Lambda(M^{m+1}) = (\Lambda X)^{\otimes m+1}$ (since the model of a product is the tensor product of the models), Δ is modelled by the $(m+1)$-fold multiplication μ and $\Lambda Y = \Lambda(T^{m+1}(M))$.

Now, however, we may make the following

Definition. The *rational category* of M (or $\Lambda(M) = \Lambda X$), $\mathrm{cat}_0(M)$, is the least m so that (**) exists; that is, there exists ρ with $\rho\xi \simeq \mu$.

Observe that: (1) $\mathrm{cat}_0(M) \leqslant \mathrm{cat}(M)$ since any diagram (*) induces a diagram (**). (2) If M is simply connected, then $\mathrm{cat}_0(M) = \mathrm{cat}(M_0)$, where M_0 is the $\mathbf{Q}$-localization of M. This follows since (*) itself localizes.

The definition of $\text{cat}_0(M)$ would be of little use if this were its only description. The passage from (*) to (**) simply transfers the difficult problem of obtaining Δ' to an (almost) equally difficult problem of obtaining ρ. However, by understanding the nature of $\Lambda Y = \Lambda(T^{m+1}(M))$, a more accessible criterion for $\text{cat}_0(M)$ may be developed. We first describe ΛY.

PROPOSITION (2.2 of [3]). *A minimal model for the fat wedge is given by a minimal model* $\phi: \Lambda Y \to \Omega$ *for the quotient* cdga

$$\Omega = (\Lambda X)^{\otimes m+1}/(\Lambda^+ X)^{\otimes m+1}$$

where $\Lambda^+ X$ *consists of all elements of positive degree. Moreover, if* $\pi: (\Lambda X)^{\otimes m+1} \to \Omega$ *is the projection, then any* $\eta: (\Lambda X)^{\otimes m+1} \to \Lambda Y$ *with* $\phi\eta \simeq \pi$ *is homotopic to the induced map* ξ.

(The existence of η is a consequence of the minimality of $(\Lambda X)^{\otimes m+1}$, the fact that ϕ induces an isomorphism of cohomology and cdga obstruction theory. See [4] or [6].)

In some sense, the form of Ω is exactly what one would expect viewing the fat wedge as a spatial bound on the "form product" length (as opposed to cuplength). The proof of the proposition relies on various technical results involving $A(T^{m+1}(M))$.

Now let $\Lambda^{>m}X$ denote the differential ideal of ΛX having additive basis the monomials $x_{i_1} \cdots x_{i_k}$ with $k > m$. Consider the projection $p: \Lambda X \to \Lambda X/\Lambda^{>m}X$ and a minimal model $\theta: \Lambda Z \to \Lambda X/\Lambda^{>m}X$. As before (for ΛY), minimal model theory provides a lift of $p, \tilde{p}: \Lambda X \to \Lambda Z$, with $\theta\tilde{p} \simeq p$.

Say that ΛX is a *retract* of $\Lambda X/\Lambda^{>m}X$ if there exists a cdga map $r: \Lambda Z \to \Lambda X$ with $r\tilde{p} \simeq 1_{\Lambda X}$.

We are now in a position to give the rational homotopy criterion for category. We give a proof in one direction and refer to [3] for the other. (Also, we make use of the fact that a cohomology isomorphism $\theta: A \to B$ induces bijections of cdga homotopy sets $\theta_*: [\Lambda, A] \xrightarrow{\simeq} [\Lambda, B]$ for any minimal Λ.) With the notation above, we have the

THEOREM. $\text{cat}_0(M) \leqslant m$ *if and only if* $\Lambda X = \Lambda(M)$ *is a retract of* $\Lambda X/\Lambda^{>m}X$.

Proof. We only prove the "if" part. Let r denote the retraction, $\Lambda Z \to \Lambda X$, with $r\tilde{p} \simeq 1_{\Lambda X}$. We have the following homotopy commutative diagram (where $\bar{\mu}$ is the map induced by μ and $\tilde{\mu}$ is a lift to models),

$$\tilde{p}\left(\begin{array}{ccc} \Lambda X & \overset{\mu}{\longleftarrow} & (\Lambda X)^{\otimes m+1} \\ \downarrow p & & \downarrow \pi \\ \dfrac{\Lambda X}{\Lambda^{>m}X} & \overset{\bar{\mu}}{\longleftarrow} & \dfrac{(\Lambda X)^{\otimes m+1}}{(\Lambda^{+}X)^{\otimes m+1}} \\ \simeq \uparrow \theta & & \simeq \uparrow \phi \\ \Lambda Z & \overset{\tilde{\mu}}{\longleftarrow} & \Lambda Y \end{array}\right)\xi$$

In order to prove $\mathrm{cat}_0(M) \leqslant m$, we must find $\rho\colon \Lambda Y \to \Lambda X$ with $\rho\xi \simeq \mu$. We can use the given retraction to do exactly this. Let $\rho = r\tilde{\mu}$.

First, observe $\theta\tilde{p}\mu \simeq p\mu = \bar{\mu}\pi \simeq \bar{\mu}\phi\xi \simeq \theta\tilde{\mu}\xi$. Because θ is a cohomology isomorphism, $\tilde{p}\mu \simeq \tilde{\mu}\xi$.

Now, $\rho\xi = r\tilde{\mu}\xi \simeq r\tilde{p}\mu \simeq 1_{\Lambda X}\mu = \mu$ and we are done. □

Of course, $\mathrm{cat}_0(M)$ is, in general, too hard to compute. However, the criterion we have described opens up the possibility of defining weaker invariants which *are* computable. In a sense, the point of this paper is to give an exposition of these weaker invariants in the context of a specific problem of interest to "geometers".

Define $e_0(M)$ to be the least integer s so that $p\colon \Lambda X \to \Lambda X/\Lambda^{>s}X$ induces an injection in cohomology. (This is, in fact, equivalent to requiring $r\colon \Lambda Z \to \Lambda X$ to be only a *linear* retraction. The invariant $e_0(M)$ was first defined by Toomer [9] in terms of the Milnor-Moore spectral sequence.)

Note that if $r\colon \Lambda Z \to \Lambda X$ is a retraction, then $\tilde{p}^*$ is injective and (since θ^* is an isomorphism) therefore so is p^*. Hence, we clearly have

$$e_0(M) \leqslant \mathrm{cat}_0(M)\ .$$

Moreover, when M is a nilpotent space (so that the full power of the minimal model may be utilized) *and* a manifold (so that Poincaré duality may be exploited), we can identify $e_0(M)$ in the following manner:

PROPOSITION. *If M^n is a nilpotent manifold with fundamental class $\tau \in H^n(M;\mathbf{Q})$, then $e_0(M)$ is the largest k such that τ is represented by a cocycle in $\Lambda^{\geqslant k}X$.*

Proof. Let $e_0(M) = s$ and let k be defined by the stated property. If τ is represented by a cocycle in $\Lambda^{>s}X$, then (for $p\colon \Lambda X \to \Lambda X/\Lambda^{>s}X$) $p^*(\tau) = 0$ and p^* is therefore not injective. Hence, $k \leqslant s$.

In order to show the reverse inequality $s \leqslant k$, we must show that, for $p: \Lambda X \to \Lambda X/\Lambda^{>k}X, p^*$ is injective. Plainly, by Poincaré duality, p^* is injective if and only if $p^*(\tau) \neq 0$. Hence, we prove this.

Suppose $p^*(\tau) = 0$. Let τ denote the representing cocycle in $\Lambda^{\geqslant k}X$ of the fundamental class τ. Let $p(\tau) = \bar{\tau} \in \Lambda X/\Lambda^{>k}X$ and consider $\bar{\tau}$ as an element in $\Lambda^{\leqslant k}X$. Now, $p^*(\tau) = 0$, so there exists $\bar{\alpha} \in \Lambda X/\Lambda^{>k}X$ with $\bar{d}\bar{\alpha} = \bar{\tau}$. Consider $\bar{\alpha} \in \Lambda^{\leqslant k}X$ as well and note that $p(d\bar{\alpha}) = \bar{d}\bar{\alpha} = \bar{\tau}$. Therefore, in ΛX

$$d\bar{\alpha} = \bar{\tau} + \Phi, \qquad \text{where } \Phi \in \Lambda^{>k}X .$$

Similarly, of course, $\tau = \bar{\tau} + \Omega$ for $\Omega \in \Lambda^{>k}X$ and we obtain,

$$\tau = \bar{\tau} + \Omega = d\bar{\alpha} - \Phi + \Omega$$

with $\Omega - \Phi \in \Lambda^{>k}X$. But this means τ is cohomologous to $\Omega - \Phi \in \Lambda^{>k}X$, contradicting the definition of k. □

§3. NILMANIFOLDS

A *nilmanifold* M is the quotient of a nilpotent Lie group N by a discrete cocompact subgroup π. The description below follows [7].

It is well known that N is diffeomorphic to some $\mathbf{R}^n$ and, therefore, M is a $K(\pi, 1)$. Furthermore, this entails the fact that π is a finitely generated torsionfree nilpotent group.

On the algebraic side, there is a refinement of the upper central series of π,

$$\pi \supseteq \pi_2 \supseteq \pi_3 \supseteq \cdots \supseteq \pi_n \supseteq 1$$

with each $\pi_i/\pi_{i+1} \cong \mathbf{Z}$ whose length is invariant and is called the *rank* of π. So, for π above, $\operatorname{rank}(\pi) = n$.

This description implies that any $u \in \pi$ has a decomposition $u = u_1^{x_1} \cdots u_n^{x_n}$, where $\langle u_n \rangle = \pi_n, \cdots \langle u_i \rangle = \pi_i/\pi_{i+1}$. The set $\{u_1, \cdots u_n\}$ is called a Malcev basis for π. Using this basis the multiplication in π takes the form

$$u_1^{x_1} \cdots u_n^{x_n} u_1^{y_1} \cdots u_n^{y_n} = u_1^{\rho_1(x,y)} \cdots u_n^{\rho_n(x,y)}$$

where

$$\rho_i(x, y) = x_i + y_i + \tau_i(x_1, \cdots x_{i-1}, y_1, \cdots y_{i-1}) .$$