

# §3. NILMANIFOLDS

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In order to show the reverse inequality  $s \leq k$ , we must show that, for  $p: \Lambda X \rightarrow \Lambda X / \Lambda^{>k} X$ ,  $p^*$  is injective. Plainly, by Poincaré duality,  $p^*$  is injective if and only if  $p^*(\tau) \neq 0$ . Hence, we prove this.

Suppose  $p^*(\tau) = 0$ . Let  $\tau$  denote the representing cocycle in  $\Lambda^{\geq k} X$  of the fundamental class  $\tau$ . Let  $p(\tau) = \bar{\tau} \in \Lambda X / \Lambda^{>k} X$  and consider  $\bar{\tau}$  as an element in  $\Lambda^{\leq k} X$ . Now,  $p^*(\tau) = 0$ , so there exists  $\bar{\alpha} \in \Lambda X / \Lambda^{>k} X$  with  $d\bar{\alpha} = \bar{\tau}$ . Consider  $\bar{\alpha} \in \Lambda^{\leq k} X$  as well and note that  $p(d\bar{\alpha}) = d\bar{\alpha} = \bar{\tau}$ . Therefore, in  $\Lambda X$

$$d\bar{\alpha} = \bar{\tau} + \Phi, \quad \text{where } \Phi \in \Lambda^{>k} X.$$

Similarly, of course,  $\tau = \bar{\tau} + \Omega$  for  $\Omega \in \Lambda^{>k} X$  and we obtain,

$$\tau = \bar{\tau} + \Omega = d\bar{\alpha} - \Phi + \Omega$$

with  $\Omega - \Phi \in \Lambda^{>k} X$ . But this means  $\tau$  is cohomologous to  $\Omega - \Phi \in \Lambda^{>k} X$ , contradicting the definition of  $k$ .  $\square$

### §3. NILMANIFOLDS

A *nilmanifold*  $M$  is the quotient of a nilpotent Lie group  $N$  by a discrete cocompact subgroup  $\pi$ . The description below follows [7].

It is well known that  $N$  is diffeomorphic to some  $\mathbf{R}^n$  and, therefore,  $M$  is a  $K(\pi, 1)$ . Furthermore, this entails the fact that  $\pi$  is a finitely generated torsionfree nilpotent group.

On the algebraic side, there is a refinement of the upper central series of  $\pi$ ,

$$\pi \supseteq \pi_2 \supseteq \pi_3 \supseteq \cdots \supseteq \pi_n \supseteq 1$$

with each  $\pi_i / \pi_{i+1} \cong \mathbf{Z}$  whose length is invariant and is called the *rank* of  $\pi$ . So, for  $\pi$  above,  $\text{rank}(\pi) = n$ .

This description implies that any  $u \in \pi$  has a decomposition  $u = u_1^{x_1} \cdots u_n^{x_n}$ , where  $\langle u_n \rangle = \pi_n, \cdots \langle u_i \rangle = \pi_i / \pi_{i+1}$ . The set  $\{u_1, \cdots, u_n\}$  is called a Malcev basis for  $\pi$ . Using this basis the multiplication in  $\pi$  takes the form

$$u_1^{x_1} \cdots u_n^{x_n} u_1^{y_1} \cdots u_n^{y_n} = u_1^{\rho_1(x, y)} \cdots u_n^{\rho_n(x, y)}$$

where

$$\rho_i(x, y) = x_i + y_i + \tau_i(x_1, \cdots, x_{i-1}, y_1, \cdots, y_{i-1}).$$

*Example.*  $N = U_n(\mathbf{R})$ , the group of upper diagonal matrices with 1's on the diagonal;  $\pi = U_n(\mathbf{Z})$ . A Malcev basis is given by  $\{u_{ij} \mid 1 \leq i < j \leq n\}$  where  $u_{ij} = I + e_{ij}$  and

$$\rho_{ij}(x, y) = x_{ij} + y_{ij} + \sum_{i < k < j} x_{ik}y_{kj}.$$

Consider the central extension  $\pi_n \rightarrow \pi \rightarrow \bar{\pi}$ . The cocycle for the extension is  $\tau_n: \bar{\pi} \times \bar{\pi} \rightarrow \mathbf{Z}$ . Of course  $\bar{\pi}$  is also finitely generated torsionfree with refined upper central series,

$$\bar{\pi} = \frac{\pi}{\pi_n} \supseteq \frac{\pi_2}{\pi_n} \supseteq \cdots \supseteq \frac{\pi_{n-1}}{\pi_n} \supseteq \frac{\pi_n}{\pi_n} = 1.$$

Hence,  $\text{rank}(\bar{\pi}) = n - 1$  and

$$\bar{\rho}_i(x, y) = \rho_i((x, 0), (y, 0)) = x_i + y_i + \tau_i(x_1, \cdots, x_{i-1}, y_1, \cdots, y_{i-1})$$

for  $i < n$ . Clearly, then, we may iterate this process and decompose  $\pi$  as  $n$  central extensions of the form

$$\mathbf{Z} \rightarrow G \rightarrow \bar{G}$$

with cocycles  $\tau_i \in H^2(\bar{G}; \mathbf{Z})$  (with untwisted coefficients since the extension is central).

This description allows a geometric formulation:

$$\tau_n \in H^2(\bar{\pi}; \mathbf{Z}) \cong H^2(K(\bar{\pi}, 1); \mathbf{Z}) \cong [K(\bar{\pi}, 1), K(\mathbf{Z}, 2)]$$

by the usual identification of cohomology groups with sets of homotopy classes into  $K(\mathbf{Z}, m)$ 's. Now,  $K(\mathbf{Z}, 2) = \mathbf{C}P(\infty)$ , the classifying space for principal  $S^1$ -bundles, so  $\tau_n$  induces a bundle over  $K(\bar{\pi}, 1)$ ,

$$\begin{array}{ccc} S^1 & \rightarrow & K(\pi, 1) \\ & & \downarrow \\ & & K(\bar{\pi}, 1) \xrightarrow{\tau_n} \mathbf{C}P(\infty). \end{array}$$

The total space of the bundle is clearly  $K(\pi, 1)$  since the ensuing short exact sequence of fundamental groups is classified by  $\tau_n$ .

Now, because we can iterate the algebraic decomposition of  $\pi$ , we obtain an iterated sequence of principal  $S^1$ -bundles classified by the  $\tau_i$ :

$$\begin{array}{ccccc}
S^1 & \rightarrow & M & = & K(\pi, 1) \\
& & \downarrow & & \\
S^1 & \rightarrow & M_{n-1} & \xrightarrow{\tau_n} & CP(\infty) \\
& & \downarrow & & \\
& & \vdots & & \\
& & \downarrow & & \\
S^1 & \rightarrow & M_1 & \xrightarrow{\tau_2} & CP(\infty) \\
& & \downarrow & & \\
& & * & \xrightarrow{\tau_1} & CP(\infty) .
\end{array}$$

We can assume (by finite dimensionality) that each  $\tau_i$  has image in a finite  $CP(n)$ , so thus may be approximated by a smooth map. Hence, each  $M_j$  is a compact manifold with

$$\dim(M_j) = \dim(M_{j-1}) + 1 .$$

Thus,  $\dim(M) = \text{rank}(\pi) = n$ .

#### §4. CATEGORY OF NILMANIFOLDS

The decomposition of  $M = K(\pi, 1)$  into a tower of principal  $S^1$ -bundles is, in fact, the Postnikov decomposition of  $M$  with  $k$ -invariants the  $\tau_i$ . By the fundamental theorem of rational homotopy theory, the minimal model has the form,

$$\Lambda(M) = (\Lambda(x_1, \dots, x_n), d) , \quad \deg(x_i) = 1$$

with  $dx_i = \tau_i$ , where  $\tau_i$  is a cocycle representing the class  $\tau_i \in H^2(M_{i-1}; \mathbf{Z})$ . Note that  $\Lambda(M)$  is an exterior algebra because all generators are in degree 1. Therefore, since  $\dim M = n$ , the only possibility for a cocycle representing the fundamental class is  $x_1 \cdots x_n$ . Hence,  $e_0(M) = n$  and this immediately implies,

*Proof of Theorem 1.*  $n = e_0(M) \leq \text{cat}_0(M) \leq \text{cat}(M) \leq \dim M = n$ .  $\square$

*Example.* Consider the 3-dimensional Heisenberg group  $U_3(\mathbf{R})$  and mod out by  $U_3(\mathbf{Z})$ . The resulting  $M$  is a 3-manifold obtained as a principal bundle,

$$S^1 \rightarrow M \rightarrow T^2$$