§3. NILMANIFOLDS

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In order to show the reverse inequality $s \leqslant k$, we must show that, for $p: \Lambda X \to \Lambda X/\Lambda^{>k}X, p^*$ is injective. Plainly, by Poincaré duality, p^* is injective if and only if $p^*(\tau) \neq 0$. Hence, we prove this.

Suppose $p^*(\tau) = 0$. Let τ denote the representing cocycle in $\Lambda^{\geqslant k}X$ of the fundamental class τ. Let $p(\tau) = \bar{\tau} \in \Lambda X/\Lambda^{>k}X$ and consider $\bar{\tau}$ as an element in $\Lambda^{\leqslant k}X$. Now, $p^*(\tau) = 0$, so there exists $\bar{\alpha} \in \Lambda X/\Lambda^{>k}X$ with $\bar{d}\bar{\alpha} = \bar{\tau}$. Consider $\bar{\alpha} \in \Lambda^{\leqslant k}X$ as well and note that $p(d\bar{\alpha}) = \bar{d}\bar{\alpha} = \bar{\tau}$. Therefore, in ΛX

$$d\bar{\alpha} = \bar{\tau} + \Phi\,, \qquad \text{where } \Phi \in \Lambda^{>k}X\,.$$

Similarly, of course, $\tau = \bar{\tau} + \Omega$ for $\Omega \in \Lambda^{>k}X$ and we obtain,

$$\tau = \bar{\tau} + \Omega = d\bar{\alpha} - \Phi + \Omega$$

with $\Omega - \Phi \in \Lambda^{>k}X$. But this means τ is cohomologous to $\Omega - \Phi \in \Lambda^{>k}X$, contradicting the definition of k. $\square$

§3. Nilmanifolds

A *nilmanifold* M is the quotient of a nilpotent Lie group N by a discrete cocompact subgroup π. The description below follows [7].

It is well known that N is diffeomorphic to some $\mathbf{R}^n$ and, therefore, M is a $K(\pi, 1)$. Furthermore, this entails the fact that π is a finitely generated torsionfree nilpotent group.

On the algebraic side, there is a refinement of the upper central series of π,

$$\pi \supseteq \pi_2 \supseteq \pi_3 \supseteq \cdots \supseteq \pi_n \supseteq 1$$

with each $\pi_i/\pi_{i+1} \cong \mathbf{Z}$ whose length is invariant and is called the *rank* of π. So, for π above, $\operatorname{rank}(\pi) = n$.

This description implies that any $u \in \pi$ has a decomposition $u = u_1^{x_1} \cdots u_n^{x_n}$, where $\langle u_n \rangle = \pi_n, \cdots \langle u_i \rangle = \pi_i/\pi_{i+1}$. The set $\{u_1, \cdots u_n\}$ is called a Malcev basis for π. Using this basis the multiplication in π takes the form

$$u_1^{x_1} \cdots u_n^{x_n} u_1^{y_1} \cdots u_n^{y_n} = u_1^{\rho_1(x,y)} \cdots u_n^{\rho_n(x,y)}$$

where

$$\rho_i(x, y) = x_i + y_i + \tau_i(x_1, \cdots x_{i-1}, y_1, \cdots y_{i-1})\,.$$

Example. $N = U_n(\mathbf{R})$, the group of upper diagonal matrices with 1's on the diagonal; $\pi = U_n(\mathbf{Z})$. A Malcev basis is given by $\{u_{ij} \mid 1 \leqslant i < j \leqslant n\}$ where $u_{ij} = I + e_{ij}$ and

$$\rho_{ij}(x, y) = x_{ij} + y_{ij} + \sum_{i<k<j} x_{ik} y_{kj} .$$

Consider the central extension $\pi_n \to \pi \to \bar{\pi}$. The cocycle for the extension is $\tau_n : \bar{\pi} \times \bar{\pi} \to \mathbf{Z}$. Of course $\bar{\pi}$ is also finitely generated torsionfree with refined upper central series,

$$\bar{\pi} = \frac{\pi}{\pi_n} \supseteq \frac{\pi_2}{\pi_n} \supseteq \cdots \supseteq \frac{\pi_{n-1}}{\pi_n} \supseteq \frac{\pi_n}{\pi_n} = 1 .$$

Hence, $\operatorname{rank}(\bar{\pi}) = n - 1$ and

$$\bar{\rho}_i(x, y) = \rho_i((x, 0), (y, 0)) = x_i + y_i + \tau_i(x_1, \cdots x_{i-1}, y_1, \cdots y_{i-1})$$

for $i < n$. Clearly, then, we may iterate this process and decompose π as n central extensions of the form

$$\mathbf{Z} \to G \to \bar{G}$$

with cocycles $\tau_i \in H^2(\bar{G}; \mathbf{Z})$ (with untwisted coefficients since the extension is central).

This desription allows a geometric formulation:

$$\tau_n \in H^2(\bar{\pi}; \mathbf{Z}) \cong H^2(K(\bar{\pi}, 1); \mathbf{Z}) \cong [K(\bar{\pi}, 1), K(\mathbf{Z}, 2)]$$

by the usual identification of cohomology groups with sets of homotopy classes into $K(\mathbf{Z}, m)$'s. Now, $K(\mathbf{Z}, 2) = \mathbf{C}P(\infty)$, the classifying space for principal S^1-bundles, so τ_n induces a bundle over $K(\bar{\pi}, 1)$,

$$\begin{array}{ccc} S^1 & \to & K(\pi, 1) \\ & & \downarrow \\ & & K(\bar{\pi}, 1) \overset{\tau_n}{\to} \mathbf{C}P(\infty) . \end{array}$$

The total space of the bundle is clearly $K(\pi, 1)$ since the ensuing short exact sequence of fundamental groups is classified by τ_n.

Now, because we can iterate the algebraic decomposition of π, we obtain an iterated sequence of principal S^1-bundles classifed by the τ_i:

$$
\begin{array}{ccccc}
S^1 & \to & M & = & K(\pi, 1) \\
 & & \downarrow & & \\
S^1 & \to & M_{n-1} & \overset{\tau_n}{\to} & \mathbf{C}P(\infty) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
S^1 & \to & M_1 & \overset{\tau_2}{\to} & \mathbf{C}P(\infty) \\
 & & \downarrow & & \\
 & & * & \overset{\tau_1}{\to} & \mathbf{C}P(\infty)\ .
\end{array}
$$

We can assume (by finite dimensionality) that each τ_i has image in a finite $\mathbf{C}P(n)$, so thus may be approximated by a smooth map. Hence, each M_j is a compact manifold with

$$\dim(M_j) = \dim(M_{j-1}) + 1\ .$$

Thus, $\dim(M) = \operatorname{rank}(\pi) = n$.

§4. Category of nilmanifolds

The decomposition of $M = K(\pi, 1)$ into a tower of principal S^1-bundles is, in fact, the Postnikov decomposition of M with k-invariants the τ_i. By the fundamental theorem of rational homotopy theory, the minimal model has the form,

$$\Lambda(M) = (\Lambda(x_1, \cdots x_n), d)\ , \qquad \deg(x_i) = 1$$

with $dx_i = \tau_i$, where τ_i is a cocycle representing the class $\tau_i \in H^2(M_{i-1}; \mathbf{Z})$. Note that $\Lambda(M)$ is an exterior algebra because all generators are in degree 1. Therefore, since $\dim M = n$, the only possibility for a cocycle representing the fundamental class is $x_1 \cdots x_n$. Hence, $e_0(M) = n$ and this immediately implies,

Proof of Theorem 1. $n = e_0(M) \leqslant \operatorname{cat}_0(M) \leqslant \operatorname{cat}(M) \leqslant \dim M = n$. □

Example. Consider the 3-dimensional Heisenberg group $U_3(\mathbf{R})$ and mod out by $U_3(\mathbf{Z})$. The resulting M is a 3-manifold obtained as a principal bundle,

$$S^1 \to M \to T^2$$