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Artikel: PERMUTATION GROUPS GENERATED BY A TRANSPOSITION AND ANOTHER ELEMENT

Kapitel: 2. An application to Galois theory

Autor: Janusz, Gerald J.

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COROLLARY 2 (C. Jordan [3]). *A primitive subgroup of* Sym(n) *containing a transposition is all of* Sym(n).

Proof. Let $\mathscr{H}$ be a primitive subgroup of Sym(n) and τ a transposition in $\mathscr{H}$. Then $\mathscr{H}$ permutes the components Γ_i of $\Gamma(\mathscr{H}, \tau)$ and so the vertex sets V_i of the Γ_i are permuted by $\mathscr{H}$. The primitivity of $\mathscr{H}$ implies that the set $\{1, 2, \cdots, n\}$ can be partitioned into disjoint subsets permuted by $\mathscr{H}$ only if each subset has order one or there is just one subset of order n. Since the vertex set of Γ_i has more than one element, there is only one component and $\mathscr{H} =$ Sym(n) by Corollary 1.

2. AN APPLICATION TO GALOIS THEORY

We extend the theorem mentioned in the introduction replacing the condition that the degree of the polynomial be a prime greater than 3 by the condition that the degree of the polynomial be divisible only by primes greater than 3.

THEOREM 2. *Left $f(x)$ be a polynomial of degree n with rational coefficients and irreducible over the rational field. Assume that $f(x)$ has exactly $n-2$ real roots. If n is divisible only by primes greater than 3 then the Galois group of the splitting field of $f(x)$ is not solvable and $f(x)$ is not solvable by radicals.*

Proof. Let $\mathscr{H}$ be the Galois group of $f(x)$ over the rational field. We view $\mathscr{H}$ as a permutation group on the n roots of f. Then complex conjugation, τ, is a transposition in $\mathscr{H}$ of the two nonreal roots. Since $f(x)$ is irreducible, $\mathscr{H}$ is transitive on the set of n roots. By theorem 1, $\mathscr{H}$ contains a subgroup isomorphic to the direct product of t copies of Sym(k) where $tk = n$. Since k is a divisor of n and $k > 1$, the hypothesis on the divisors of n implies $k \geqslant 5$. Thus Sym(k) is not a solvable group and $\mathscr{H}$ is not solvable as it contains a nonsolvable subgroup. Thus $f(x)$ is not solvable by radicals.

3. TWO GENERATOR SUBGROUPS OF Sym(n)

Next we apply Theorem 1 to determine the subgroup of Sym(n) generated by a transposition and one other element. We first consider the case in which