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COROLLARY 2 (C. Jordan [3]). *A primitive subgroup of  $\text{Sym}(n)$  containing a transposition is all of  $\text{Sym}(n)$ .*

*Proof.* Let  $\mathcal{H}$  be a primitive subgroup of  $\text{Sym}(n)$  and  $\tau$  a transposition in  $\mathcal{H}$ . Then  $\mathcal{H}$  permutes the components  $\Gamma_i$  of  $\Gamma(\mathcal{H}, \tau)$  and so the vertex sets  $V_i$  of the  $\Gamma_i$  are permuted by  $\mathcal{H}$ . The primitivity of  $\mathcal{H}$  implies that the set  $\{1, 2, \dots, n\}$  can be partitioned into disjoint subsets permuted by  $\mathcal{H}$  only if each subset has order one or there is just one subset of order  $n$ . Since the vertex set of  $\Gamma_i$  has more than one element, there is only one component and  $\mathcal{H} = \text{Sym}(n)$  by Corollary 1.

## 2. AN APPLICATION TO GALOIS THEORY

We extend the theorem mentioned in the introduction replacing the condition that the degree of the polynomial be a prime greater than 3 by the condition that the degree of the polynomial be divisible only by primes greater than 3.

**THEOREM 2.** *Let  $f(x)$  be a polynomial of degree  $n$  with rational coefficients and irreducible over the rational field. Assume that  $f(x)$  has exactly  $n - 2$  real roots. If  $n$  is divisible only by primes greater than 3 then the Galois group of the splitting field of  $f(x)$  is not solvable and  $f(x)$  is not solvable by radicals.*

*Proof.* Let  $\mathcal{H}$  be the Galois group of  $f(x)$  over the rational field. We view  $\mathcal{H}$  as a permutation group on the  $n$  roots of  $f$ . Then complex conjugation,  $\tau$ , is a transposition in  $\mathcal{H}$  of the two nonreal roots. Since  $f(x)$  is irreducible,  $\mathcal{H}$  is transitive on the set of  $n$  roots. By theorem 1,  $\mathcal{H}$  contains a subgroup isomorphic to the direct product of  $t$  copies of  $\text{Sym}(k)$  where  $tk = n$ . Since  $k$  is a divisor of  $n$  and  $k > 1$ , the hypothesis on the divisors of  $n$  implies  $k \geq 5$ . Thus  $\text{Sym}(k)$  is not a solvable group and  $\mathcal{H}$  is not solvable as it contains a nonsolvable subgroup. Thus  $f(x)$  is not solvable by radicals.

## 3. TWO GENERATOR SUBGROUPS OF $\text{Sym}(n)$

Next we apply Theorem 1 to determine the subgroup of  $\text{Sym}(n)$  generated by a transposition and one other element. We first consider the case in which