COMPLEX GROWTH SERIES OF COXETER SYSTEMS

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1. INTRODUCTION

Let (W, S) be a *Coxeter system* (see [1] for the definitions). Throughout this paper the generating set S of W is assumed to be finite. S determines a length on W called *word length*. It is defined by

$$l(w) = l_S(w) = \min\{r \mid w = s_1 \dots s_r, s_i \in S\} ,$$

for $w \in W$. The *growth series* of W with respect to S is the formal series

$$W_S(t) = \sum_{w \in W} t^{l(w)} .$$

For a subset $X \subseteq S$, we denote by W_X the subgroup of W generated by X; the system (W_X, X) is still a Coxeter system.

With a Coxeter system (W, S) one can associate a simplicial complex $\Sigma(W, S)$, called the *Coxeter complex*. This was introduced by Tits in [5] and is an essential ingredient of the theory of buildings (see [2] and [6]).

In this paper we introduce a new formal series $W_S(t_1, t_2)$, in two variables, which will be called the *complex growth series* of (W, S), and is determined from the complex $\Sigma(W, S)$. More precisely,

$$W_S(t_1, t_2) = \sum_F t_1^{d(C_0, F)} t_2^{\operatorname{codim}(F)} ,$$

where the sum is over all the faces F of $\Sigma(W, S)$ (here we assume the empty set to be a face of $\Sigma(W, S)$ of dimension -1), and $d(C_0, F)$ is the distance between the fundamental chamber C_0 of $\Sigma(W, S)$ and the face F.

The notions of Coxeter complex, chamber, face and fundamental chamber will be recalled in Section 2.

MAIN THEOREM. *Let (W, S) be a Coxeter system. Then*

$$W_S(t_1, t_2) = \sum_{X \subseteq S} t_2^{|X|} \frac{W_S(t_1)}{W_X(t_1)} . \tag{1.1}$$

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Solomon proved in [4] that, if (W, S) is a finite Coxeter system (i.e. W is finite), then

$$(1.2) \qquad \sum_{X \subseteq S} (-1)^{|X|} \frac{W_S(t)}{W_X(t)} = t^m ,$$

where

$$m = \max_{w \in W} l(w) .$$

Later, in [1, §4.1, exercise 26], Bourbaki proved a similar formula for an infinite system (W, S); in that case,

$$(1.3) \qquad \sum_{X \subseteq S} (-1)^{|X|} \frac{W_S(t)}{W_X(t)} = 0 .$$

Several results on growth series of Coxeter groups are obtained by induction on $|S|$ using (1.2) and (1.3). We refer to [3] for an exposition on those two equalities and their applications.

An immediate corollary of (1.1), (1.2) and (1.3) is: if (W, S) is a finite Coxeter system, then

$$(1.4) \qquad W_S(t_1, -1) = t_1^m ,$$

m being the maximal length in W; and if (W, S) is an infinite Coxeter system, then

$$(1.5) \qquad W_S(t_1, -1) = 0 .$$

In fact, these two equalities (1.4) and (1.5) can and will be proved independently of the formulas (1.1), (1.2) and (1.3) (Proposition 2).

As an illustration of the Main Theorem, let us give two explicit examples.

1) Assume

$$W = < s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = 1 >$$

to be a Coxeter group of type A_2. The geometric realisation of $\Sigma(W, S)$ is an hexagon. We have

$$W_S(t_1, t_2) = (1 + t_1)(1 + t_1 + t_1^2) + 2(1 + t_1 + t_1^2)t_2 + t_2^2 ,$$

and

$$W_{\{s_1, s_2\}}(t) = (1 + t)(1 + t + t^2) ,$$
$$W_{\{s_1\}}(t) = W_{\{s_2\}}(t) = 1 + t,$$
$$W_\emptyset(t) = 1.$$

Thus the equality (1.1) holds in that case.

2) Assume

$$W = < s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_1 s_3)^3 = (s_2 s_3)^3 = 1 >$$

to be a Coxeter group of type $\tilde{A}_2$. The geometric realisation of $\Sigma(W, S)$ is a plane. We have

$$W_S(t_1, t_2) = \frac{(1+t_1)(1+t_1+t_1^2)}{(1-t_1)(1-t_1^2)} + 3\frac{1+t_1+t_1^2}{(1-t_1)(1-t_1^2)} t_2$$
$$+ 3\frac{1}{(1-t_1)(1-t_1^2)} t_2^2 + t_2^3,$$

and

$$W_S(t) = \frac{(1+t)(1+t+t^2)}{(1-t)(1-t^2)},$$
$$W_{\{s_1, s_2\}}(t) = W_{\{s_1, s_3\}}(t) = W_{\{s_2, s_3\}}(t) = (1+t)(1+t+t^2),$$
$$W_{\{s_1\}}(t) = W_{\{s_2\}}(t) = W_{\{s_3\}}(t) = 1 + t,$$
$$W_\emptyset(t) = 1.$$

Thus the equality (1.1) holds in that case.

In Section 2 we will recall some definitions in the theory of Coxeter complexes, we will define the complex growth series of a Coxeter system (W, S), we will prove that $W_S(t_1, 0) = W_S(t_1)$ and $W_S(0, t_2) = (1 + t_2)^{|S|}$ (Proposition 1), we will prove the equalities (1.4) and (1.5) (Proposition 2), and we will prove the Main Theorem.

2. Complex growth series

We assume the reader to be familiar with the notions of simplicial complex, chamber complex, adjacency between two chambers, gallery and labelling. We refer to [2, Chap. I, Appendix] for a good exposition of these notions.

Let (W, S) be a Coxeter system. A *special coset* of (W, S) is a coset wW_X, with $w \in W$ and $X \subseteq S$. We denote by $\Sigma = \Sigma(W, S)$ the poset of all special cosets, ordered by the reverse inclusion; $B \leqslant A$ in Σ if $B \supseteq A$ in W. The poset Σ is a labelled chamber simplicial complex (see [2, Chap. III, § 1]).

A *chamber* of Σ is a singleton $\{w\}$ with $w \in W$. A *vertex* of Σ is a special coset $wW_{S-\{s\}}$ with $w \in W$ and $s \in S$. The face of Σ of dimension -1 is the coset $1 . W = W$ (this face has 0 vertices). The *fundamental chamber* of Σ is $\{1\}$.

The Coxeter group W naturally acts on Σ by

$$w(vW_X) = (wv)\,W_X\,, \tag{2.1}$$

where $w \in W$, and vW_X is a face of Σ (i.e. a special coset).

The map which associates to a face $F = wW_X$ the subset $\lambda(F) = S - X$ of S determines a labelling on Σ, called the *canonical labeling* of Σ, where $\lambda(F)$ is the *type* of a face F.

Two chambers $\{w\} \neq \{w'\}$ are *adjacent* if they have a common codimension 1 face, namely, if there exists an $s \in S$ such that $w' = ws$. A *gallery of length d* is a sequence $\{C_i\}_{i=0}^{d}$ of $d+1$ chambers such that C_i and C_{i+1} are adjacent for $i = 0, 1, \dots, d-1$. In fact, to give a gallery $\{C_i\}_{i=0}^{d}$ is equivalent to give a source chamber C_0 and a sequence $s_1, \dots, s_d$ of elements of S; the equivalence is given by $C_i = s_i \dots s_2 s_1 (C_0)$. A gallery $\{C_i\}_{i=1}^{d}$ joining two chambers C_0 and C_d is called *minimal* if there is no gallery joining C_0 and C_d with a smaller length.

The *distance* $d(C, D)$ between two chambers C and D is the length of a minimal gallery joining C and D. We can easily see that, if $C = \{w\}$ and $D = \{v\}$, then

$$d(C, D) = l(w^{-1}v)\,. \tag{2.2}$$

The *distance* $d(C, F)$ between a chamber C and a face F of Σ is

$$d(C, F) = \min\{d(C, D) \mid D \text{ a chamber having } F \text{ as face}\}\,. \tag{2.3}$$

As in (2.2), if $C = \{w\}$ and $F = vW_X$, then

$$d(C, F) = \min\{l(u) \mid u \in w^{-1}vW_X\}\,. \tag{2.4}$$

The *complex growth series* of a Coxeter system (W, S) is the formal series in two variables

$$W_S(t_1, t_2) = \sum_F t_1^{d(C_0, F)}\, t_2^{\operatorname{codim}(F)}\,, \tag{2.5}$$

where the sum is over all the faces F of Σ, and where $C_0 = \{1\}$ is the fundamental chamber.

Before stating and proving Propositions 1 and 2 and the Main Theorem, we are going to state two known results (Lemmas 1 and 2). A proof of Lemma 1 can be found either in [1, §4.1, exercise 3] or in [3, Lemma 1]. A proof of Lemma 2 can be found in [2, Chap. IV, §6].

Let $X \subseteq S$ be a subset and let $v \in W$. The element v is called *X-minimal* if v is of minimal length among the elements of vW_X.

LEMMA 1. *Let* $X \subseteq S$ *be a subset and let* $v \in W$ *be an X-minimal element of* W. *Then*

i) v *is the unique X-minimal element of* vW_X,

ii) *for every* $w = vu \in vW_X$, *with* $u = v^{-1}w \in W_X$, *one has* $l(w) = l(v) + l(u)$.

For an integer $d \geqslant 0$, we denote by Σ_d the subcomplex of $\Sigma = \Sigma(W, S)$ generated by the chambers C of Σ at distance $\leqslant d$ of $C_0 = \{1\}$.

$$\Sigma_d = \bigcup_F F ,$$

where the union is over all the faces F of Σ such that $d(C_0, F) \leqslant d$. We denote by $|\Sigma_d|$ the geometric realization of Σ_d.

LEMMA 2. i) *Let* (W, S) *be a finite Coxeter system. Set* $m = \max_{w \in W} l(w)$. *Then* $|\Sigma_d|$ *is contractible if* $d < m$, *and* $|\Sigma_d|$ *is homotopic to the sphere* $\mathbf{S}^{|S|-1}$ *of dimension* $|S| - 1$ *if* $d \geqslant m$.

ii) *Let* (W, S) *be an infinite Coxeter system. Then* $|\Sigma_d|$ *is contractible.*

PROPOSITION 1. *Let* (W, S) *be a Coxeter system. Then*

$$W_S(t_1, 0) = W_S(t_1) \quad \textit{and} \tag{2.6}$$

$$W_S(0, t_2) = (1 + t_2)^{|S|} . \tag{2.7}$$

Proof.

$$W_S(t_1, 0) = \sum_F t_1^{d(C_0, F)} ,$$

where the sum is over all the faces of Σ of codimension 0, i.e. over all the chambers of Σ. Furthermore, if $F = C = \{w\}$, then, by (2.2), $d(C_0, F) = l(w)$. It follows that

$$W_S(t_1, 0) = \sum_{w \in W} t_1^{l(w)} = W_S(t_1) .$$

Now,

$$W_S(0, t_2) = \sum_F t_2^{\operatorname{codim}(F)} ,$$

where the sum is over all the faces F of Σ at distance 0 of C_0, i.e. over all the faces of C_0. Since C_0 is an $|S| - 1$ dimensional simplex, it has $\binom{|S|}{i}$ faces of dimension i (where $i = 0, 1, \ldots, |S|$). It follows that

$$W_S(0, t_2) = \sum_{i=0}^{|S|} \binom{|S|}{i} t_2^i = (1 + t_2)^{|S|} . \qquad \square$$

PROPOSITION 2. i) *Let* (W, S) *be a finite Coxeter system. Then*

$$W_S(t_1, -1) = t_1^m , \tag{2.8}$$

where m *is the maximal length in* W.

ii) *Let* (W, S) *be an infinite Coxeter system. Then*

$$W_S(t_1, -1) = 0 . \tag{2.9}$$

Proof. Recall that Σ_d is the subcomplex of Σ generated by the chambers of Σ at distance $\leqslant d$ of C_0, and that $|\Sigma_d|$ is the geometric realization of Σ_d. We denote by $E(|\Sigma_d|)$ the Euler characteristic of $|\Sigma_d|$. It is well known that $E(|\Sigma_d|)$ can be computed as follows:

$$(-1)^{|S|-1} E(|\Sigma_d|) = (-1)^{|S|-1} \sum_{\substack{d(C_0, F) \leqslant d \\ F \neq W}} (-1)^{\dim(F)}$$

$$= \sum_{\substack{d(C_0, F) \leqslant d \\ F \neq W}} (-1)^{\operatorname{codim}(F)} .$$

On the other hand, we have

$$W_S(t_1, -1) = \sum_F t_1^{d(C_0, F)} (-1)^{\operatorname{codim}(F)}$$

$$= \sum_{d=0}^{\infty} \left(\sum_{d(C_0, F) = d} (-1)^{\operatorname{codim}(F)} \right) t_1^d .$$

Thus

$$(-1)^{|S|-1} (E(|\Sigma_d|) - E(|\Sigma_{d-1}|)) \tag{2.10}$$

is the coefficient of t_1^d in $W_S(t_1, -1)$ for $d \geqslant 1$, and

$$(-1)^{|S|-1} E(|\Sigma_0|) + (-1)^{|S|} \tag{2.11}$$

is the coefficient of t_1^0 in $W_S(t_1, -1)$. Lemma 2 implies that, if (W, S) is a finite Coxeter system, then

$$E(|\Sigma_d|) = \begin{cases} 1 & \text{if} \quad d < m , \\ 1 + (-1)^{|S|-1} & \text{if} \quad d \geqslant m , \end{cases}$$

where m is the maximal length in W; and if (W, S) is an infinite Coxeter system, then

$$E(|\Sigma_d|) = 1 ,$$

for all $d \geqslant 0$. Replacing $E(|\Sigma_d|)$ by its value in (2.10) and (2.11), we obtain the equalities (2.8) and (2.9). □

MAIN THEOREM. *Let* (W, S) *be a Coxeter system. Then*

$$W_S(t_1, t_2) = \sum_{X \subseteq S} t_2^{|X|} \frac{W_S(t_1)}{W_X(t_1)} . \tag{2.12}$$

Proof. Recall that the map which associates to a face $F = wW_X$ the subset $\lambda(F) = S - X$ of S determines a labelling on Σ, where $\lambda(F)$ is the type of the face F. Clearly, if $\lambda(F) = Y$, then $\dim(F) = |Y| - 1$ and $\operatorname{codim}(F) = |S| - |Y| = |S - Y|$. Therefore

$$W_S(t_1, t_2) = \sum_{Y \subseteq S} t_2^{|S - Y|} \left(\sum_{F \in \mathcal{F}_Y} t_1^{d(C_0, F)} \right) , \tag{2.13}$$

where $\mathcal{F}_Y$ is the set of faces of Σ of type Y. Let us prove

$$\sum_{F \in \mathcal{F}_Y} t_1^{d(C_0, F)} = \frac{W_S(t_1)}{W_{S-Y}(t_1)} , \tag{2.14}$$

for every $Y \subseteq S$. The equalities (2.13) and (2.14) clearly imply (2.12).

Let $X = S - Y$. Recall that an element $v \in W$ is X-minimal if it is of minimal length in vW_X. Every face $F \in \mathcal{F}_Y$ can be written $F = vW_X$ with v X-minimal (take any element of minimal length in F). By (2.4), we have

$$d(C_0, F) = l(v) .$$

Lemma 1 shows that, for every $F \in \mathcal{F}_Y$, there is an unique X-minimal element v in F. Therefore

$$\sum_{F \in \mathcal{F}_Y} t_1^{d(C_0, F)} = \sum_{v \in A_X} t_1^{l(v)} , \tag{2.15}$$

where A_X is the set of all the X-minimal elements of W. Finally, Lemma 1 shows

$$\begin{aligned} W_S(t_1) &= \sum_{w \in W} t_1^{l(w)} \\ &= \sum_{v \in A_X} \sum_{w \in vW_X} t_1^{l(w)} \quad \text{(Lemma 1.i)} \\ &= \sum_{v \in A_X} \sum_{u \in W_X} t_1^{l(v)+l(u)} \quad \text{(Lemma 1.ii)} \\ &= \left(\sum_{v \in A_X} t_1^{l(v)} \right) W_X(t_1) \,. \end{aligned}$$

This and (2.15) imply (2.14) □

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