B. Her work

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forcefully apparent to me than in the last stormy summer, that of 1933, which we spent together in Göttingen... A time of struggle like this one... draws people closer together; thus I have a particularly vivid recollection of these months. Emmy Noether, her courage, her frankness, her unconcern about her own fate, her conciliatory spirit, were in the midst of all the hatred and meanness, despair and sorrow surrounding us, a moral solace... The memory of her work in science and of her personality among her fellows will not soon pass away. She was a great mathematician, the greatest, I firmly believe, that her sex has ever produced, and a great woman.

B. HER WORK

I will now give an account of some of Emmy Noether's major contributions to mathematics, indicating their sources.

Irving Kaplansky called her the "mother of modern algebra" ([23], p. 155). Saunders MacLane asserted that "abstract algebra, as a conscious discipline, starts with Emmy Noether's 1921 paper 'Ideal Theory in Rings'" ([28], p. 10). Hermann Weyl claimed that she "changed the face of algebra by her work" ([41], p. 128). It is a tall order to try to do justice to these assertions, but let me try.

According to van der Waerden, the essence of Emmy Noether's mathematical credo is contained in the following maxim ([5], p. 42):

> All relations between numbers, functions and operations become perspicuous, capable of generalization, and truly fruitful after being detached from specific examples, and traced back to conceptual connections.

We identify these ideas with the abstract, axiomatic approach in mathematics. They sound commonplace to us. But they were not so in Emmy Noether's time. In fact, they are commonplace today in considerable part *because* of her work.

Algebra in the 19th century was concrete by our standards. It was connected in one way or another with real or complex numbers. For example, some of the great contributors to algebra in the 19th century, mathematicians whose works shaped the algebra of the 20th century, were Gauss, Galois, Jordan, Kronecker, Dedekind, and Hilbert. Their algebraic works dealt with quadratic forms, cyclotomy, field extensions, permutation groups, ideals in rings of integers of algebraic number fields, and invariant theory. All of these works were related in one way or another to real or complex numbers.

Moreover, even these important works in algebra were viewed in the 19th century, in the overall mathematical scheme, as secondary. The primary mathematical fields in that century were analysis (complex analysis, differential equations, real analysis), and geometry (projective, noneuclidean, differential, and algebraic). But after the work of Emmy Noether and others in the 1920s, algebra became central in mathematics.

It should be noted that Emmy Noether was not the only, nor even the only major, contributor to the abstract, axiomatic approach in algebra. Among her predecessors who contributed to the genre were Cayley and Frobenius in group theory, Dedekind in lattice theory, Weber and Steinitz in field theory, and Wedderburn and Dickson in the theory of hypercomplex systems. Among her contemporaries, Albert in the U.S. and Artin in Germany stand out.

The "big bang" theory rarely applies when dealing with the origin of mathematical ideas. So also in Emmy Noether's case. The concepts she introduced and the results she established must be viewed against the background of late-19th-and early-20th-century contributions to algebra. She was particularly influenced by the works of Dedekind. In discussing her contributions she frequently used to say, with characteristic modesty: "It can already be found in Dedekind's work" ("Es steht schon bei Dedekind") ([12], p. 68). In commenting on them, I will thus be considering their roots in Dedekind's work and in that of others from which she drew inspiration and on which she built.

Emmy Noether contributed to the following major areas of algebra: invariant theory (1907-1919), commutative algebra (1920-1929), noncommutative algebra and representation theory (1927-1933), and applications of noncommutative algebra to problems in commutative algebra (1932-1935). She thus dealt with just about the whole range of subject-matter of the algebraic tradition of the 19th and early 20th centuries (with the possible exception of group theory proper). What is significant is that she transformed that subject-matter, thereby originating a new algebraic tradition — what has come to be known as modern or abstract algebra.

I will now discuss Emmy Noether's contributions to each of the above areas.

INVARIANT THEORY

Emmy Noether's statement (quoted above), that her ideas are already in Dedekind's work, could, with equal validity, have been put as "It all started with Gauss". Indeed, invariant theory dates back to Gauss' study of binary quadratic forms in his *Disquisitiones Arithmeticae* of 1801. Gauss defined an equivalence relation on such forms and showed that the discriminant is an invariant of the form under equivalence (see [1]). A second important source of invariant theory is projective geometry, which originated in the 1820s. A significant problem was to distinguish euclidean from projective properties of geometric figures. The projective properties turned out to be those invariant under "projective transformations" (see [26], [31]).

Formally, invariant theory began with Cayley and Sylvester in the late 1840s. Cayley used it to bring to light the deeper connections between metric and projective geometry (see [10]). Although important connections with geometry were maintained throughout the 19th and early 20th centuries, invariant theory soon became an area of investigation independent of its relations to geometry. In fact, it became an important branch of *algebra* in the second half of the 19th century. To Sylvester "all algebraic inquiries, sooner or later, end at the Capitol of modern algebra over whose shining portal is inscribed the Theory of Invariants" ([26], p. 930).

An important problem of the abstract theory of invariants was to discover invariants of various "forms".[1]) Many of the major mathematicians of the second half of the 19th century worked on the computation of invariants of specific forms. This led to the major problem of invariant theory, namely to determine a complete system of invariants (a basis) for a given form; i.e., to find invariants of the form — it was conjectured that finitely many would do — such that every other invariant could be expressed as a combination of these. Cayley showed in 1856 that the finitely many invariants he had found earlier for binary quartic forms (i.e., forms of degree four in two variables) are a complete system. About ten years later Gordan proved that every binary form (of any degree) has a finite basis. Gordan's proof of this important result was computational – he *exhibited* a complete system of invariants.[2]) In 1888 Hilbert astonished the mathematical world by announcing a new, conceptual, approach to the problem of invariants. The idea was to consider, instead of invariants, expressions in a finite number of variables, in short, the polynomial ring in those variables. Hilbert then proved what came to be

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[1]) E.g., a *binary form* is an expression of the form $f(x_1, x_2) = a_0x_1^n + a_1x_1^{n-1}x_2 + \dots + a_nx_2^n$. If this form is transformed by a linear transformation T of the variables x_1 and x_2 into the form $F(X_1, X_2) = A_0X_1^n + A_1X_1^{n-1}X_2 + \dots + A_nX_2^n$, then any function I of the coefficients of f which satisfies the relation $I(A_0, \dots, A_n) = r^kI(a_0, \dots, a_n)$ is called an *invariant* of f under T (r denotes the determinant of T).

[2]) Weyl observed that "there exist papers of his [Gordan's] where twenty pages of formulas are not interrupted by a single word; it is told that in all his papers he himself wrote the formulas only, the text being added by his friends" ([41], p. 117).

known as the Basis Theorem, namely that every ideal in the ring of polynomials in finitely many variables has a finite basis. A corollary was that every form (of any degree, in any number of variables) has a finite complete system of invariants. Gordan's reaction to Hilbert's proof, which did not explicitly exhibit the complete system of invariants, was that "this is not mathematics; it is theology" ([26], p. 930).[1])

Emmy Noether's thesis, written under Gordan in 1907, was entitled "On Complete Systems of Invariants for Ternary Biquadratic Forms". The thesis was computational, in the style of Gordan's work. It ended with a table of the complete system of 331 invariants for such a form. Noether was later to describe her thesis as "a jungle of formulas" ([24], p. 11).[2])

Emmy Noether obtained, however, several notable results on invariants during the 1910s. First, using the methods she had developed in two papers (in 1915 and 1916) on the subject, she made a significant contribution to the problem, first posed by Dedekind, of finding a Galois extension of a given number field with a prescribed Galois group.[3]) Second, during her work in Göttingen on differential invariants, she used the calculus of variations to obtain the so-called Noether Theorem, still important in mathematical physics (see [7], p. 125). The physicist Fez Gursey says of this contribution ([22], p. 23):

> The key to the relation of symmetry laws to conservation laws in physics is Emmy Noether's celebrated theorem which states that a dynamical system described by an action under a Lie group with n parameters admits n invariants (conserved quantities) that remain constant in time during the evolution of the system.

Alexandrov summarizes her work on invariants by noting that it "would have been enough... to earn her the reputation of a first class mathematician" ([2], p. 156).

What was the route that led Emmy Noether from the computational theory of invariants to the abstract theory of rings and modules?[4]) In 1910 Gordan retired from the University of Erlangen and was soon replaced by Ernst

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[1]) Later Hilbert gave a constructive proof of his result which, however, he did not consider significant, but which elicited from Gordan the statement: "I have convinced myself that theology also has its advantages" ([26], p. 930).

[2]) When asked in 1932 to review a paper on invariants, she refused, declaring "I have completly forgotten all of the symbolic calculations I ever learned" ([12], p. 18).

[3]) The problem, in this generality, is still unresolved, although it has been solved for symmetric and solvable groups (see [7], p. 115).

[4]) "A greater contrast is hardly imaginable than between her first paper, the dissertation, and her works of maturity", remarks Weyl ([41], p. 120).

Fischer. He, too, was a specialist in invariant theory, but invariant theory of the Hilbert persuasion. Emmy Noether came under his influence and gradually made the change from Gordan's algorithmic approach to invariant theory to Hilbert's conceptual approach. Later work on invariants brought her in contact with the famous joint paper of Dedekind and Weber (see p. 115 below) on the arithmetic theory of algebraic functions. She became "sold" on Dedekind's approach and ideas, and this determined the direction of her future work.

COMMUTATIVE ALGEBRA

The two major sources of commutative algebra are algebraic geometry and algebraic number theory. Emmy Noether's two seminal papers of 1921 and 1927 on the subject can be traced, respectively, to these two sources. In these papers, entitled, respectively, Ideal Theory in Rings (Idealtheorie in Ringbereichen) and Abstract Development of Ideal Theory in Algebraic Number Fields and Function Fields (Abstrakter Aufbau der Idealtheorie in algebraischen Zahl-und Funktionenkörpern), she broke fundamentally new ground, originating "a new and epoch-making style of thinking in algebra" ([41], p. 130).

Algebraic geometry had its origins in the study, begun in the early 19th century, of abelian functions and their integrals. This analytic approach to the subject gradually gave way to geometric, algebraic, and arithmetic means of attack. In the algebraic context, the main object of study is the ring of polynomials $k[x_1, x_2, \ldots, x_n]$, k a field (in the 19th century k was the field of real or complex numbers). Hilbert in the 19th century, and Lasker and Macauley in the early 20th century, had shown that in such a ring every ideal is a finite intersection of primary ideals, with certain uniqueness properties.[1]) (Geometrically, the result says that every variety is a unique, finite, union of irreducible varieties.) In her 1921 paper Emmy Noether generalized this result to arbitrary commutative rings with the ascending chain condition (a.c.c.).[2]) Her main result was that in such a ring every ideal is a finite intersection (with accompanying uniqueness properties) of primary ideals. (See [14] for historical and [3] for technical details.)

What was so significant about this paper which (we recall) MacLane singled out as marking the beginning of abstract algebra as a conscious discipline?

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[1]) An ideal I in a commutative ring R is called *primary* if $xy \in I$ implies $x \in I$ or $y^t \in I$ for some positive integer t. The concept of primary ideal is an extension to rings of prime power for the integers.

[2]) A commutative ring R satisfies the *ascending chain condition* if every ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ terminates; i.e., $I_n = I_{n+1} = \ldots$ for some positive integer n.

First and foremost was the isolation of the a.c.c. as the crucial concept needed in the proof of the main result. In fact, the proof "rested entirely on elementary consequences of the chain condition and... [was] startling in... simplicity" ([22], p. 13). Earlier proofs (of the corresponding result for polynomial rings) involved considerable computation, such as elimination theory and the geometry of algebraic sets.

The a.c.c. did not originate with Emmy Noether. Dedekind (in 1894) and Lasker (in 1905) used it, but in concrete settings of rings of algebraic integers and of polynomials, respectively. Moreover, the a.c.c. was for them incidental rather than of major consequence. Noether's isolation of the a.c.c. as an important concept was a watershed. Thanks to her work, rings with the a.c.c., now called noetherian rings [1]), have been singled out for special attention. In fact, commutative algebra has been described as the study of (commutative) noetherian rings. As such, the subject had its formal genesis in Emmy Noether's 1921 paper.

Another fundamental concept with Emmy Noether highlighted in the 1921 paper is that of a ring. This concept, too, did not originate with her. Dedekind (in 1871) introduced it as a subset of the complex numbers closed under addition, substraction, and multiplication, and called it an "order". Hilbert (in 1897), in his famous Report on Number Theory (Zahlbericht), coined the term "ring", but only in the context of rings of integers of algebraic number fields. Fraenkel (in 1914) gave essentially the modern definition of ring, but postulated two extraneous conditions. Noether (in the 1921 paper) gave the definition in current use (given also, apparently, by Sono in 1917, but this went unnoticed).

But it was not merely Noether's *definition* of the concept of ring which proved important. Through her groundbreaking papers in which the concept of ring played an essential role (and of which the 1921 paper was an important first), she brought this concept into prominence as a central notion of algebra. It immediately began to serve as the starting point for much of abstract algebra, taking its rightful place alongside the concepts of group and field, already reasonably well established at that time.

Noether also began to develop in the 1921 paper a general theory of ideals for commutative rings. Notions of prime, primary, and irreducible ideal, of intersection and product of ideals, of congruence modulo an ideal — in short, much of the machinery of ideal theory, appears here.

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[1]) A term coined in 1943 by Chevalley.

Toward the end of the paper she defined the concept of *module* over a noncommutative ring and showed that some of the earlier decomposition results for ideals carry over to submodules. (I will discuss modules in connection with Noether's work in noncommutative algebra.)

To summarize, the 1921 paper introduced and gave prominence to what came to be some of the basic concepts of abstract algebra, namely ring, module, ideal, and the a.c.c. Beyond that, it introduced, and began to show the efficacy of, a new way of doing algebra — abstract, axiomatic, conceptual. No mean accomplishment for a single paper! (See [19] and [22] for further details.)

Emmy Noether's 1927 paper had its roots in algebraic number theory and, to a lesser extent, in algebraic geometry. The sources of algebraic number theory are Gauss' theory of quadratic forms of 1801, his study of biquadratic reciprocity of 1832 (in which he introduced the Gaussian integers), and attempts in the early 19th century to prove Fermat's Last Theorem. In all cases the central issue turned out to be unique factorization in rings of integers of algebraic number fields.[1]) When examples of such rings were found in which unique factorization fails,[2]) the problem became to try to "restore", in some sense, the "paradise lost". This was achieved by Dedekind in 1871 (and, in a different way, by Kronecker in 1882) when he showed that unique factorization can be reestablished if one considers factorization of ideals (which he had introduced for this purpose) rather than of elements. His main result was that if R is the ring of integers of an algebraic number field, then every ideal of R is a unique product of prime ideals.[3]) (See [6] for historical and [34] for technical details.)

Riemann introduced "Riemann surfaces" in the 1850s in order to facilitate the study of (multivalued) algebraic functions. His methods were, however, nonrigorous, and depended on physical considerations. In 1882 Dedekind and Weber wrote an all-important paper whose aim was to give rigorous, algebraic, expression to some of Riemann's ideas on complex

[1]) An *algebraic number field* is a finite extension of the rationals, $Q(\alpha) = \{a_0 + a_1\alpha + \dots + a_n\alpha^n : a_i \in Q, \alpha \text{ an algebraic number}\}$. The *ring of integers* of $Q(\alpha)$ consists of the elements of $Q(\alpha)$ which are roots of *monic* polynomials with *integer* coefficients. See [1] for details.

[2]) $R = \{a + b\sqrt{-5} : a, b \in \mathbf{Z}\}$ is such an example. Here

$$6 = 2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

are two distinct decomposition of 6 as a product of primes of R.

[3]) An ideal I of a ring R is said to be *prime* if $xy \in I$ implies $x \in I$ or $y \in I$. Prime ideals are generalizations of primes in the ring of integers.

function theory, in particular to his notion of a Riemann surface. Their idea was to establish an analogy between algebraic number fields and algebraic function fields, and to carry over the machinery and results of the former to the latter. They succeeded admirably, giving (among other things) a purely algebraic definition of a Riemann surface, and an algebraic proof of the fundamental Riemann-Roch Theorem. At least as importantly, they pointed to what proved to be a most fruitful idea, namely the interplay between algebraic number theory and algebraic geometry.

More specifically, just as in algebraic number theory one associates an algebraic number field $Q(\alpha)$ with a given algebraic number, so in algebraic geometry one associates an algebraic function field $\mathbf{C}(x, y)$ with a given algebraic function. $\mathbf{C}(x, y)$ consists of polynomials in x and y with complex coefficients, where y satisfies a polynomial equation with coefficients in $\mathbf{C}(x)$ (i.e., y is algebraic over $\mathbf{C}(x)$).[1] If A is the "ring of integers" of $\mathbf{C}(x, y)$ (i.e., A consists of the roots in $\mathbf{C}(x, y)$ of *monic* polynomials with coefficients in $\mathbf{C}[x]$), then a major result of the Dedekind-Weber paper is that every ideal in A is a unique product of prime ideals. (See [14] and [26] for historical, and [9] and [16] for technical, details.)

In her 1927 paper Emmy Noether generalized the above decomposition results for algebraic number fields and function fields to commutative rings. In fact, she characterized those commutative rings in which every ideal is a unique product of prime ideals. Such rings are now called *Dedekind domains.* She showed that R is a Dedekind domain if and only if (1) R statisfies the a.c.c., (2) R/I statisfies the d.c.c. for every nonzero ideal I of R, (3) R is an integral domain (i.e., it has an identity and no zero divisors), and (4) R is integrally closed in its field of quotients. Condition (4) proved particularly significant since it singled out the basic notion of integral dependence (related to that of integral closure).[2]) This concept (already present in Dedekind's work on algebraic numbers) has proved to be of fundamental importance in commutative algebra. As Gilmer notes, "the concept of integral dependence is to *Aufbau* [Noether's 1927 paper] what the a.c.c. is to *Idealtheorie* [her 1921 paper]" ([19], p. 136). Among other basic results she proved in this paper are: (a) the (by now standard) isomorphism and homomorphism theorems for rings and modules, (b) that a module M has a composition series if and only if it

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[1]) $\mathbf{C}(x, y)$ is an extension field of $\mathbf{C}$ of transcendence degree 1; i.e., x is transcendental over $\mathbf{C}$ and y is algebraic over $\mathbf{C}(x)$. Thus, in analogy with the algebraic number field $Q(\alpha)$, $\mathbf{C}(x)$ corresponds to Q and y to α.

[2]) Let $R \subseteq S$ be rings. An element $s \in S$ is *integrally dependent* on R (or is integral over R) if it satisfies a monic polynomial with coefficients in R. R is *integrally closed* in S if every element of S which is integral over R belongs to R.

satisfies both the a.c.c. and d.c.c., (c) that if an R-module M is finitely generated and R satisfies the a.c.c. (d.c.c.), then so does M.

To summarize Emmy Noether's contributions to commutative algebra: in addition to proving important results, she introduced concepts and developed techniques which have become standard tools of the subject. In fact, her 1921 and 1927 papers, combined with those of Krull of the 1920s, are said to have created the subject of commutative algebra.

NONCOMMUTATIVE ALGEBRA AND REPRESENTATION THEORY

Before her ideas in commutative algebra had been fully assimilated by her contemporaries, Emmy Noether turned her attention to the other major algebraic subjects of the 19th and early 20th centuries, namely hypercomplex number systems (what we now call associative algebras) and groups (in particular, group representations). She extended and unified these two subjects through her abstract, conceptual approach, in which module-theoretic ideas that she had used in the commutative case played a crucial role.

The theory of hypercomplex systems began with Hamilton's 1843 introduction of the quaternions. At the end of the 19th century, E. Cartan, Frobenius, and Molien gave structure theorems for such systems over the real and complex numbers, and in 1907 Wedderburn extended these to hypercomplex systems over arbitrary fields. In the spirit of Emmy Noether's work in commutative algebra, Artin extended Wedderburn's results to (noncommutative, semisimple) rings with the descending chain condition. (See [25] for details.)

Groups were the first algebraic systems to be developed extensively. By the end of the 19th century they began to be studied abstractly. An important tool in that study was representation theory, developed by Burnside, Frobenius, and Molien in the 1890s (see [20]). The idea was to study, instead of the abstract group, its concrete representations in terms of matrices (A *representation* of a group is a homomorphism of the group into the group of invertible matrices of some given order.)

In her 1929 paper Hypercomplex Numbers and Representation Theory (Hyperkomplexe Grössen und Darstellungstheorie) Emmy Noether framed group representation theory in terms of the structure theory of hypercomplex systems. The main tool in this approach was the *module*. The idea was to associate with each representation ϕ of G by invertible matrices with entries in some field k, a $k(G)$-module V called the *representation module* of ϕ($k(G)$ is the *group algebra* of G over k). Conversely, any $k(G)$-module M gives rise to a representation ψ of G.[1]) This establishes a one-one correspondence between representations of G (over k) and $k(G)$-modules. The standard concepts of representation theory can now be phrased in terms of modules. For example, two representations are equivalent if and only if their representation modules are isomorphic; a representation is irreducible if and only if its representation module is simple. The techniques of module theory, and the structure theory of hypercomplex systems (applied to the hypercomplex system $k(G)$) can now be used to "recast the foundations of group representation theory" ([27], p. 150). (See [27] for historical and [11] for technical details.)

Noether's work in this area created a very effective conceptual framework in which to study representation theory. For example, while the (computational) classical approach to representation theory is valid only over the field of complex numbers (or, at best, over an algebraically closed field of characteristic 0), Noether's approach remains meaningful for any field (of any characteristic). The use of general fields in representation theory became important in the 1930s when Brauer began his pioneering studies of *modular representations* (i.e., those in which the characteristic of the field divides the order of the group). Noether's ideas also "planted the seed of modern integral representation theory" ([27], p. 152), that is, representation theory over commutative rings rather than over fields. Noether herself extended the representation theory of groups to that of semi-simple artinian rings; here she needed the concept of a *bimodule.*

A word about modules, which were so central in Emmy Noether's work in both commutative and noncommutative algebra. Dedekind, in connection with his 1871 work in algebraic number theory, was the first to use the term "module", but to him it meant a subgroup of the additive group of complex numbers (i.e., a Z-module); in 1894 he developed an extensive theory of such modules. Lasker, in his 1905 work on decomposition of polynomial rings, used the terms "module" and "ideal" interchangeably (the former he applied to polynomial rings over $\mathbf{C}$, the latter to such rings over $\mathbf{Z}$). Noether was the first to use the notion of module abstractly (with a ring as domain of operators) and to recognize its potential. In fact, it is through her work that the concept of module became the central concept of algebra that it is today. Indeed, modules are important not only because of their unifying, but also because of their *linearizing,* power. (They are, after all, generalizations of vector

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[1]) In one direction, consider ϕ as a homomorphism of G into $L(V, V)$, the set of linear transformations of a vector space V over k. We turn V into a $k(G)$-module by defining $v \cdot g = \phi(g)(v)$, for $v \in V$, $g \in G$, and extending by linearity to all of $k(G)$. In the other direction, define $\psi: G \to L(M, M)$ by $\psi(g)(m) = m \cdot g$. See [11], Chapter II, for details.

spaces, and many of the standard vector-space constructions, such as subspace, quotient space, direct sum, and tensor product carry over to modules.)[1]) In fact, the importance of the invention of *homological algebra* was that it carried the process of linearization far forward by developing tools for its implementation. (E.g., the functors "Ext" and "Tor" measure the extent to which modules over general rings "misbehave" when compared to modules over fields, viz. vector spaces; see [8].)

Applications of noncommutative to commutative algebra

Noether believed that the theory of noncommutative algebras is governed by simpler laws than that of commutative algebra. In her 1932 plenary address at the International Congress of Mathematicians in Zurich, entitled Hypercomplex Systems and their Relations to Commutative Algebra and Number Theory (Hyperkomplexe Systeme in ihren Beziehungen zur kommutativen Algebra und Zahlentheorie), she outlined a program putting that belief into practice. Her program has been called "a foreshadowing of modern cohomology theory" ([35], p. 8). The ideas on factor sets contained therein were soon used by Hasse and Chevalley "to obtain some of the main results on global and local class field theory" ([22], p. 26). Noether's own immediate objective was to apply the theory of central simple algebras (as developed by her, Brauer, and others) to problems in class field theory. (See [7], [35], and [36].)

Some of her ideas (and those of others) on the interplay between commutative and noncommutative algebra had already recently born fruit with the proof of the celebrated Albert-Brauer-Hasse-Noether Theorem. This result, called by Jacobson "one of the high points of the theory of algebras" ([22], p. 21), gives a complete description of finite-dimensional division algebras over algebraic number fields.[2]) It is important in the study of finite-dimensional algebras and of group representations.

To bring out the context of the above theorem, it should be noted that Wedderburn's 1907 structure theorems for finite-dimensional algebras reduced their study to that of nilpotent algebras and division algebras. Since the unravelling of the structure of the former seemed (and still seems, despite considerable progress) "hopeless", attention focussed on the latter.

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[1]) We know the power of linearization in analysis. Modules can be said to provide analogous power in algebra.

[2]) They are intimately related to the "cyclic" algebras studied earlier by Dickson (see [21], Vol. II, p. 480).

Considerable progress on the structure of division algebras was made in the late 1920s and early 1930s. The Albert-Brauer-Hasse-Noether Theorem was a high point of these researches. It should be stressed, however, that even today much is still unknown about finite-dimensional division algebras.

C. Her legacy

The concepts Emmy Noether introduced, the results she obtained, and the mode of thinking she promoted, have become part of our mathematical culture. As Alexandrov put it ([2], p. 158):

> It was she who taught us to think in terms of simple and general algebraic concepts — homomorphic mappings, groups and rings with operators, ideals — and not in cumbersome algebraic computations; and [she] thereby opened up the path to finding algebraic principles in places where such principles had been obscured by some complicated special situation...

Moreover, as Weyl noted, "her significance for algebra cannot be read entirely from her own papers; she had great stimulating power and many of her suggestions took shape only in the works of her pupils or co-workers" ([41], pp. 129-130). Indeed, Weyl himself acknowledged his indebtedness to her in his work on groups and quantum mechanics. Among others who have *explicitly* mentioned her influence on their algebraic works are Artin, Deuring, Hasse, Jacobson, Krull, and Kurosh.

Another important vehicle for the spread of Emmy Noether's ideas was the now-classic treatise of van der Waerden entitled "Modern Algebra", first published in 1930. (It was based on lectures of Noether and Artin — see [39].) Its wealth of beautiful and powerful ideas, brilliantly presented by van der Waerden, has nurtured a generation of mathematicians. The book's immediate impact is poignantly described by Dieudonné and G. Birkhoff, respectively:

> I was working on my thesis at that time; it was 1930 and I was in Berlin. I still remember the day that van der Waerden came out on sale. My ignorance in algebra was such that nowadays I would be refused admittance to a university. I rushed to those volumes and was stupefied to see the new world which opened before me. At that time my knowledge of algebra went no further than *mathématiques spéciales,* determinants, and a little on the solvability of equations and unicursal curves. I had graduated from the École Normale and I did not know what an ideal was, and only just knew what a group was! This gives you an idea of what a young French mathematician knew in 1930 ([13], p. 137).