SIMPLE PROOF OF A THEOREM OF THUE ON THE MAXIMAL DENSITY OF CIRCLE PACKINGS IN E^2

L'Enseignement Mathématique, t. 38 (1992), p. 125-131

A SIMPLE PROOF OF A THEOREM OF THUE ON THE MAXIMAL DENSITY OF CIRCLE PACKINGS IN E^2

by Wu-Yi HSIANG

INTRODUCTION

The classical circle packing problem is to find out how densely a large number of identical circles can be packed together. In the limiting case of infinite expanse, one seeks the *maximal density* that can be achieved by all possible circle packings of the whole Euclidean plane E^2. A simple basic fact in circle packing is that a circle can be surrounded by *six kissing circles* in a unique, tight arrangement. Intuitively, this is clearly the *tightest local circle packing* and it is also easy to see that this type of tight local packing can, in fact, be infinitely repeated to fill the whole plane. Therefore, it is rather natural to expect that the above regular, hexagonal type of circle packing will be the *densest* possible circle packing. A proof of the above expected *maximality* of the density of the hexagonal circle packing was first given by Thue in 1910 [Thu]. In this short note, we shall give another proof of the above interesting basic fact of plane geometry which is simple, elementary and short.

LOCAL CELL AND LOCAL DENSITY

To each given circle Γ_0 in a given packing $\mathcal{P}$, it is quite natural to associate a surrounding region which consists of those points that are as close to its center as to the center of any other. We shall call it the *local cell of* Γ_0 *in* $\mathcal{P}$ and denote it by $C(\Gamma_0, \mathcal{P})$. The *local density* of $\mathcal{P}$ at Γ_0 is defined to be the ratio between the areas of the circle and its surrounding local cell. For example, the local cell of any circle in the above hexagonal regular packing is always a *circumscribing regular hexagon*. Therefore, it is easy to see that the local density of the above packing at any circle is equal to $\pi/\sqrt{12} = 0.906899682...$. Observe that the *(global) density* of a packing $\mathcal{P}$ is clearly just a weighted average of the local densities of its individual circles, a universal upper bound of the local density is automatically also an upper bound of the global density. Therefore, the proof of Thue's theorem on the *maximality of the global density* of the hexagonal regular circle packing can be reduced to the proof of the *maximality of the local density* of the local hexagonal circle surrounding, namely

THEOREM. *The optimal universal upper bound for the local density of circle packing in* E^2 *is equal to* $\pi/\sqrt{12}$ *and it can be realized as the local density when and only when the local cell is a circumscribing regular hexagon.*

Proof. Let Γ_0 be an arbitrary circle in a given circle packing $\mathscr{P}$, $N(\Gamma_0)$ be the set of neighboring circles whose local cells have common edges with the local cell of Γ_0 and $\hat{N}(\Gamma_0)$ be the subset of $N(\Gamma_0)$ whose centers are within a distance of 2.30 times the radii. We shall call $N(\Gamma_0)$ the set of *neighbors* of Γ_0 and $\hat{N}(\Gamma_0)$ the set of *close neighbors* of Γ_0.

Choose the center of Γ_0 to be the origin and the common radii to be the unit of length. Let O_j be the center of $\Gamma_j \in \hat{N}(\Gamma_0)$ and set A_j to be the intersection point of $\overline{OO_j}$ and Γ_0. In case that both $\overline{OO_j}$ and $\overline{OO_{j+1}}$ reach the upper limit of 2.30, $\overline{A_jA_{j+1}}$ is larger than or equal to 2/2.30 and hence the angular separation $\theta_j = \widehat{A_jA_{j+1}}$ is at least

$$2 \operatorname{Arcsin}\left(\frac{1}{2.30}\right) = 0.89959372 > \frac{2\pi}{7}. \tag{1}$$

Since the base angles of the isosceles triangle ΔOO_jO_{j+1} is considerably smaller than $\pi/2$, namely, $\operatorname{Arccos}\left(\frac{1}{2.30}\right) = 1.120999466$, the angular separation, $\widehat{A_j\ A_{j+1}}$, will always be greater than the above $2 \operatorname{Arcsin}\left(\frac{1}{2.30}\right)$ if one or both center distances are less than 2.30. Therefore, there can be at most six *close* neighbors.

Case 1: Suppose that all the neighbors are close neighbors, namely, $N(\Gamma_0) = \hat{N}(\Gamma_0)$. Let $\{\theta_j; 1 \leqslant j \leqslant n\}$ be the angular separations between the adjacent A's and $T\{A_j\}$ be the circumscribing n-gon bounded by the n tangent lines at the A's. Then, it is easy to see that $T\{A_j\}$ is always a subset of the local cell $C(\Gamma_0, \mathscr{P})$ and the area of $T\{A_j\}$ is given by

$$\sum_{j=1}^{n} \tan \frac{\theta_j}{2}, \quad \sum \frac{\theta_j}{2} = \pi, \quad \frac{\pi}{7} < \frac{\theta_j}{2} < \frac{\pi}{2}. \tag{2}$$

Now, it follows easily from the convexity of the function tan x that

$$\sum_{j=1}^{n} \tan \frac{\theta_j}{2} \geqslant n \tan \frac{\pi}{n}, \quad n \leqslant 6. \tag{3}$$

Therefore the area of $C(\Gamma_0, \mathscr{P})$ is at least equal to $6 \tan \frac{\pi}{6} = 2\sqrt{3}$ and it is equal to $2\sqrt{3}$ when and only when $C(\Gamma_0, \mathscr{P})$ is itself a circumscribing regular hexagon.

Case 2: Suppose that $N(\Gamma_0) \neq \hat{N}(\Gamma_0)$, namely, there is at least one neighboring circle with center distance exceeding 2.30. Let Γ' be such a neighbor of Γ_0.

Let us first consider the most critical situation that Γ' touches two close neighbors, say Γ_1 and Γ_2, which are actually *touching* neighbors of Γ_0. Then the geometry of the above four touching circles is represented as in Figure 1 where

$$\overline{OV} = \sec \frac{\theta_1}{2}, \quad \overline{OH} = 2 \cos \frac{\theta_1}{2}, \quad \overline{HB_1} = \cot \frac{\theta_1}{2}\, \overline{VH} \tag{4}$$

and the intersection of $C(\Gamma_0, \mathscr{P})$ and the angular region of $\theta_1 = \angle A_1OA_2$ is the pentagon $OA_1B_1B_2A_2$. Since it is assumed that $\overline{OV} > \overline{OH} > 1.15$, it follows from (4) that θ_1 lies between $\pi/2$ and $2 \operatorname{Arccos} 0.575 = 1.916384358$. Moreover, the area of the quadrilateral OA_1VA_2 is equal to $\tan \frac{\theta_1}{2}$ and the area of ΔVB_2B_1 is equal to $\overline{VH} \cdot \overline{HB_1}$ and it follows from (4) that

$$\begin{aligned} \overline{VH} \cdot \overline{HB_1} &= \cot \frac{\theta_1}{2}\, \overline{VH}^2 = \cot \frac{\theta_1}{2} \left(\sec \frac{\theta_1}{2} - 2 \cos \frac{\theta_1}{2} \right)^2 \\ &= \frac{\left(2 \cos^2 \frac{\theta_1}{2} - 1 \right)^2}{\cos \frac{\theta_1}{2} \cdot \sin \frac{\theta_1}{2}} = 2 \frac{\cos^2 \theta_1}{\sin \theta_1}. \end{aligned} \tag{5}$$

Therefore, the area of the pentagon $OA_1B_1B_2A_2$ is given by

$$\hat{A}(\theta_1) = \tan \frac{\theta_1}{2} - 2 \frac{\cos^2 \theta_1}{\sin \theta_1}, \quad \frac{\pi}{2} < \theta_1 < 1.916384358. \tag{6}$$

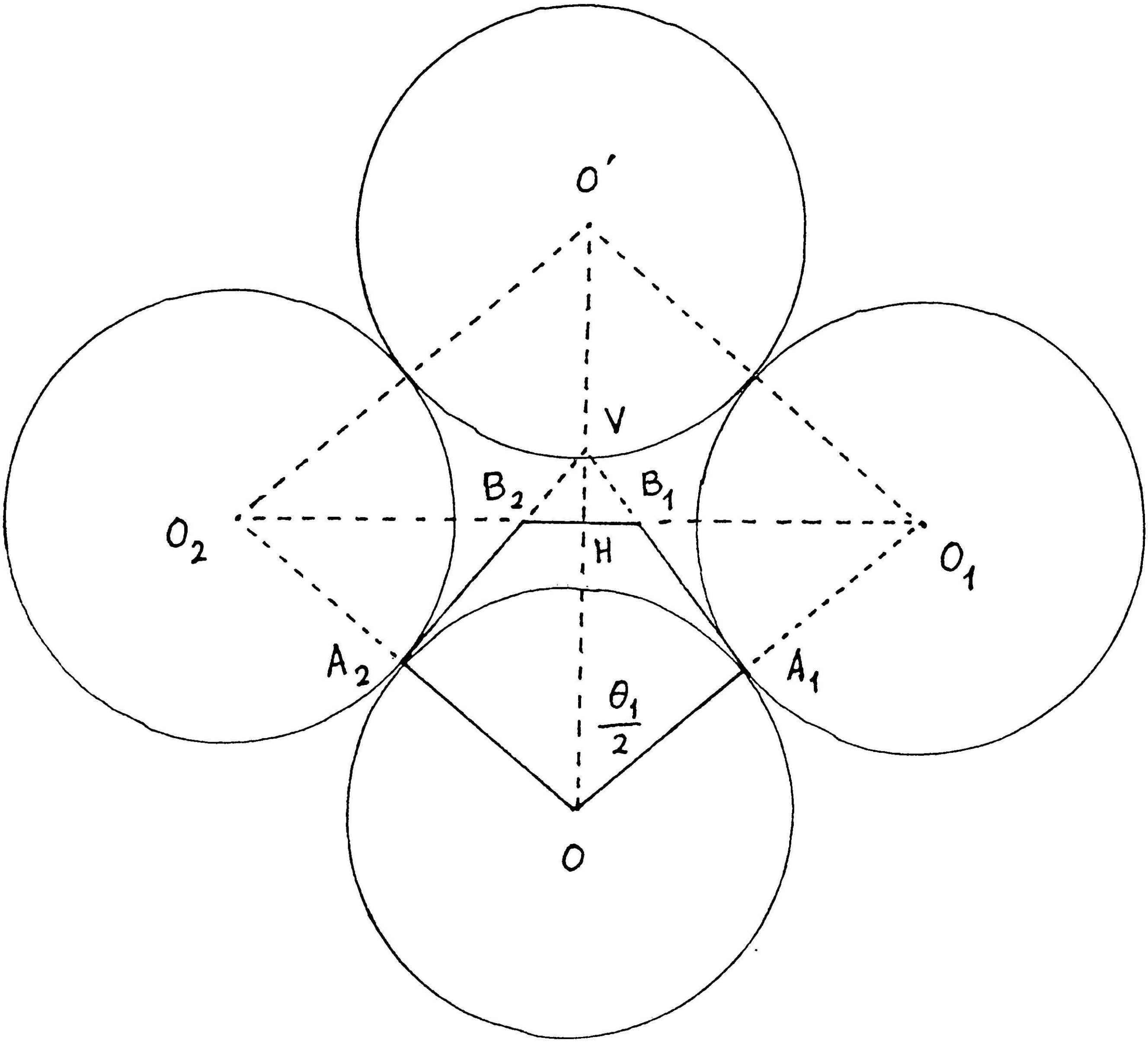

FIGURE 1

Set

$$\psi(\theta) = \tan\frac{\theta}{2} - \frac{\theta}{2} - 2\,\frac{\cos^2\theta}{\sin\theta}\,,\quad \frac{\pi}{2} < \theta < 1.917\,. \tag{7}$$

Then

$$\psi'(\theta) = \frac{1}{2}\tan^2\frac{\theta}{2} + 2\,\frac{\cos\theta}{\sin^2\theta}\,(1+\sin^2\theta)$$

$$= \frac{1}{2\sin^2\theta}\,\{(1-\cos\theta)^2 + 4\cos\theta(2-\cos^2\theta)\} \tag{8}$$

$$= \frac{1}{2\sin^2\theta}\,\{1 + 6u + u^2 - 4u^3\}$$

where u lies between $\cos(1.917)$ and 0. From (8), it is easy to show that $\psi'(\theta)$ has exactly one root θ_0 in the above range of $[\pi/2, 1.917]$, namely, $\psi'(\theta) > 0$ (resp. < 0) for $\frac{\pi}{2} \leqslant \theta < \theta_0$ (resp. $\theta_0 < \theta < 1.917$). Therefore

$$(9) \qquad \psi(\theta) = \hat{A}(\theta) - \frac{\theta}{2} \geqslant \min\left\{\psi\left(\frac{\pi}{2}\right), \psi(1.917)\right\} = \psi\left(\frac{\pi}{2}\right) = 1 - \pi/4 = 0.214601836 .$$

Therefore, if there are at least two non-close neighbors, then the above estimate already implies that the area of the local cell $C(\Gamma_0, \mathscr{P})$ must be more than $\pi + 0.42 > 2\sqrt{3}$.

Finally, let us consider the remaining case that there is exactly one non-close neighbor of Γ_0. If the number of close neighbors of Γ_0 is less than 6, then the proof of Case 1 also applies to $N(\Gamma_0, \mathscr{P})$ instead of $\hat{N}(\Gamma_0, \mathscr{P})$. If the number of close neighbors of Γ_0 is equal to 6, then the area of $C(\Gamma_0, \mathscr{P})$ is clearly bounded below by

$$(10) \qquad \hat{A}(\theta_1) + \sum_{j=2}^{6} \tan\frac{\theta_j}{2} \geqslant \hat{A}(\theta_1) + 5\tan\frac{2\pi - \theta_1}{10}$$

where θ_1 may assume to be between $\frac{\pi}{2}$ and 1.92 without loss of generality. It follows from (9) that $\hat{A}(\theta_1) - \frac{\theta_1}{2} > 0.2146$ and it is easy to see that

$$(11) \quad 5\tan\frac{2\pi - \theta_1}{10} - \left(\pi - \frac{\theta_1}{2}\right) \geqslant 5\tan\frac{2\pi - 1.917}{10} - (\pi - 0.9585) > 0.15$$

and hence

$$\hat{A}(\theta_1) + 5\tan\frac{2\pi - \theta_1}{10} > \pi + 0.3646 > 2\sqrt{3} .$$

This completes the proof of the theorem and hence also the theorem of Thue that $\pi/\sqrt{12}$ is indeed the optimal upper bound of global density of circle packings in E^2.

REMARK ON THE UNIQUENESS OF THE FINITE PACKINGS OF MAXIMAL DENSITY

All natural or practical examples of circle packings such as bees living in a honeycomb or a bundle of fibre-glass optical tubes are always packing problems of a finite number of circles (i.e. packings of their cross-section circles). The infinite circle packings of the entire plane of E^2 are actually the limit situation of the finite circle packings. Therefore, it is natural to give an appropriate definition of the concept of global density for a finite circle packing. We propose the following definition of a cluster of circles and the (global) density of a cluster of circles, namely

Definition. A packing of finite number of equal circles is called a *cluster of circles* if any two of them can be linked through neighboring pairs of center distances less than $2\sqrt{2}$ times the radii.

Let $\mathscr{C}$ be a given cluster of circles. Then, an extension, $\mathscr{C}^*$, of $\mathscr{C}$ is called a *saturated coating of* $\mathscr{C}$ if all circles of $\mathscr{C}^* \setminus \mathscr{C}$ are neighbors of some circles in $\mathscr{C}$ and it is impossible to add any more such neighbors to $\mathscr{C}^*$. Observe that *every circle* in $\mathscr{C}$ has a *saturated set of neighbors in* $\mathscr{C}^*$ and hence has a well-defined *local cell* with respect to $\mathscr{C}^*$. The *usual weighted average* of all the local densities of circles in $\mathscr{C}$ with respect to the given saturated coating $\mathscr{C}^*$ is defined to be the density of $\mathscr{C}$ in $\mathscr{C}^*$, i.e. $\rho(\mathscr{C} \text{ rel } \mathscr{C}^*)$.

Definition: The global density of $\mathscr{C}$ is defined to be the least upper bound of the densities of $\mathscr{C}$ in all possible saturated coatings of $\mathscr{C}$, namely

$$\rho(\mathscr{C}) = \text{l.u.b.}\{\rho(\mathscr{C} \text{ rel } \mathscr{C}^*)\}$$

where $\mathscr{C}^*$ run through all possible saturated coatings of $\mathscr{C}$.

UNIQUENESS THEOREM (On finite circle packings of maximal density). *$\pi/\sqrt{12}$ is still the maximal possible global density of all clusters of circles, and the global density of a cluster of circles, $\mathscr{C}$, attains the above maximum of $\pi/\sqrt{12}$ when and only when $\mathscr{C}$ is a subcluster of circles in the hexagon packing.*

Proof. It is again a direct consequence of the above Theorem on the maximal local density and its uniqueness.

REFERENCES

[S-M] SEGRE, B. and K. MAHLER. On the densest packing of circles. *Amer. Math. Montly 51* (1944), 261-270.

[Thue] THUE, A. Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene. *Norske Vid Selsk Skr. No. 1* (1910), 1-9.

(Reçu le 15 septembre 1991)

Wu-Yi Hsiang

Department of Mathematics
University of California
Berkeley, CA 94720 (USA)

vide-leer-empty