Zeitschrift: L'Enseignement Mathématique

Band: 38 (1992)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

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Kapitel: Introduction

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DOI: https://doi.org/10.5169/seals-59487

L'Enseignement Mathématique, t. 38 (1992), p. 125-131

A SIMPLE PROOF OF A THEOREM OF THUE ON THE MAXIMAL DENSITY OF CIRCLE PACKINGS IN E^2

by Wu-Yi Hsiang

Introduction

The classical circle packing problem is to find out how densely a large number of identical circles can be packed together. In the limiting case of infinite expanse, one seeks the *maximal density* that can be achieved by all possible circle packings of the whole Euclidean plane E^2. A simple basic fact in circle packing is that a circle can be surrounded by *six kissing circles* in a unique, tight arrangement. Intuitively, this is clearly the *tightest local circle packing* and it is also easy to see that this type of tight local packing can, in fact, be infinitely repeated to fill the whole plane. Therefore, it is rather natural to expect that the above regular, hexagonal type of circle packing will be the *densest* possible circle packing. A proof of the above expected *maximality* of the density of the hexagonal circle packing was first given by Thue in 1910 [Thu]. In this short note, we shall give another proof of the above interesting basic fact of plane geometry which is simple, elementary and short.

Local cell and local density

To each given circle Γ_0 in a given packing $\mathscr{P}$, it is quite natural to associate a surrounding region which consists of those points that are as close to its center as to the center of any other. We shall call it the *local cell of* Γ_0 *in* $\mathscr{P}$ and denote it by $C(\Gamma_0, \mathscr{P})$. The *local density* of $\mathscr{P}$ at Γ_0 is defined to be the ratio between the areas of the circle and its surrounding local cell. For example, the local cell of any circle in the above hexagonal regular packing is always a *circumscribing regular hexagon*. Therefore, it is easy to see that the local density of the above packing at any circle is equal to $\pi/\sqrt{12} = 0.906899682\ldots$. Observe that the *(global) density* of a packing $\mathscr{P}$ is clearly just a weighted average of the local densities of its individual circles,