4. Continued Fractions for Algebraic Numbers

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Some early papers discussing the distribution of partial quotients include Gyldén [123, 124]; Brodén [45]; and Wiman [302].

For the classical metric theory of continued fractions, see (in addition to the papers mentioned above) Lévy [187, 188, 189, 191]; Khintchine [162, 163]; and Denjoy [83, 84, 85]. For more recent improvements, see Szüsz [289, 290]; de Vroedt [296]; Wirsing [303]; Rieger [261]; Babenko [12]; and Babenko and Jur'ev [13].

A more modern approach derives these results using powerful methods of ergodic theory. A good introduction is the book of Billingsley [31]. Other articles include Knopp [168]; Doeblin [91]; Ryll-Nardzewski [266]; Hartman, Marczewski, and Ryll-Nardzewski [137]; Hartman [136]; Lévy [190]; Rényi [257]; de Vroedt [297]; Stackelberg [283]; Šalát [267]; Philipp [239, 240, 241, 242, 243]; Philipp and Stackelberg [244]; and Galambos [112, 113, 114].

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A major open problem is to determine if any algebraic numbers of degree > 2 are in $\mathscr{B}$. As Khintchine [164, 165, 160] has remarked,

> It is interesting to note that we do not, at the present time, know the continued-fraction expansion of a single algebraic number of degree higher than 2. We do not know, for example, whether the sets of elements [partial quotients] in such expansions are bounded or unbounded. In general, questions connected with the continued-fraction expansion of algebraic numbers of higher degree than the second are extremely difficult and have hardly been studied.

(The problem goes back at least to 1949, with the appearance of Khintchine's book [164]. The paragraph above most likely also appeared in the first (1936) edition of Khintchine's book, but I have not been able to verify this by examining a copy. I do not know any earlier explicit reference to the problem. A remark similar to Khintchine's was made by Delone in a foreword to a translation of Delone and Fadeev [82, p. iv].)

Khintchine's remark is still true today; there are only a few papers that have explicitly discussed the partial quotients of algebraic numbers of degree > 2. See, for example, Davenport[1]) [76]; Orevkov [229]; Pass [231]; Wolfskill [304]; Blinov and Rabinovich [34]; Bombieri and van der Poorten [37]; Dzenskevich and Shapiro [98]; and van der Poorten [247].

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[1]) Actually, Davenport's results apply to all irrational numbers, not just algebraic numbers. Also see Mendès France [206].

One can deduce weak upper bounds on the growth of the partial quotients of algebraic numbers from results in Diophantine approximation. Suppose there exist constants C, s such that

$$\| q\theta \| > \frac{C}{q^s}$$

for all positive integers q. Wolfskill [304] remarked that the partial quotients a_i in the continued fraction expansion of θ then satisfy $a_i < A^{(s+\varepsilon)^i}$, where A depends on C and ε. Thus upper bounds can be deduced from the results in the following papers: Liouville [192]; Thue [291]; Siegel [280, 281]; Dyson [97]; Roth [265]; Davenport and Roth [77]; Baker [18, 19]; Feldman [105]; Bombieri [35]; Bombieri and Mueller [36]; Chudnovsky [55]; Easton [99]; and Baker and Stewart [22]. Stronger results were given by Davenport and Roth [77]. They showed that the denominators q_i of convergents to a real algebraic number θ satisfy

$$\log\log q_n < \frac{Cn}{\sqrt{\log n}};$$

here C is a constant that depends on θ but not on n. Furthermore, this constant can be made effective.

There are several methods known for computing the partial quotients for a given algebraic number. See the papers of Lagrange [177]; Vincent [294]; Cantor, Galyean, and Zimmer [50]; Churchhouse [56]; Rosen and Shallit [264]; Akritas and Ng [6, 7]; Thull [292]; and Akritas [1, 2, 3, 4, 5].

In 1769, Lagrange [177] showed that the real zero of $x^3 - 2x - 5$ has a continued fraction expansion which begins

$$[2, 10, 1, 1, 2, 1, 3, 1, 1, 12, \ldots].$$

For some other explicit computations of the continued fraction expansions of algebraic numbers of degree > 2, see von Neumann and Tuckerman [217]; Richtmyer, Devaney, and Metropolis [260]; Bryuno [46]; Lang and Trotter [181]; Richtmyer [259]; and Pethö [238]. In 1964, J. Brillhart found that the real zero of $x^3 - 8x - 10$ had some unusually large partial quotients. An explanation was provided later by Churchhouse and Muir [57] and Stark [284].