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Artikel: REAL NUMBERS WITH BOUNDED PARTIAL QUOTIENTS: A SURVEY
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Then Graham and van Lint [119] proved the following theorem:

$$\limsup_{n\to\infty} n\delta_\theta(n) < \infty$$

if and only if θ is a number of constant type.

Boyd and Steele [43] introduced the function $l_n^+(\theta)$, the length of the longest increasing subsequence of $\{\theta\}, \{2\theta\}, \dots, \{n\theta\}$. They proved that

$$\liminf_{n\to\infty} \frac{l_n^+(\theta)}{\sqrt{n}} > 0$$

and

$$\limsup_{n\to\infty} \frac{l_n^+(\theta)}{\sqrt{n}} < \infty$$

if and only if the partial quotients of θ are bounded.

For some other results on $\{n\theta\}$ connected with bounded partial quotients, see Ennola [100, 101]; Lesca [185]; Drobot [92]; and Strauch [288].

12. Discrepancy and Dispersion

Let $\omega = (x_1, x_2, x_3, \dots)$ be a sequence of real numbers. Let $I \subseteq [0, 1)$ be an interval and let $|I|$ denote its length. Define the counting function $S_n(I) = S_n(I, \omega)$ as the number of terms x_k, $1 \leqslant k \leqslant n$, for which $\{x_k\} \in I$.

The *discrepancy* $D_n(x_1, x_2, \dots, x_n)$ is a measure of how much the sequence $x_1, x_2, \dots, x_n$ deviates from a uniform distribution. It is defined as follows:

$$D_n(\omega) = D_n(x_1, x_2, \dots, x_n) = \sup_{I \subseteq [0,1)} \left| \frac{S_n(I, \omega)}{n} - |I| \right|.$$

Now consider the discrepancy of the sequence $\omega = (\theta, 2\theta, 3\theta, \dots)$. If θ has bounded partial quotients, then the discrepancy of ω is small. In particular, we have the following estimate: If $K(\theta) \leqslant k$, then

$$nD_n(\omega) \leqslant 3 + \left(\frac{1}{\log\alpha} + \frac{k}{\log(k+1)}\right)\log n$$

for $\alpha = \frac{1}{2}(1 + \sqrt{5})$. See, for example, Kuipers and Niederreiter [173].

For other results connecting discrepancy and the boundedness of the partial quotients, see the papers of Niederreiter [218] and Dupain and Sós [94, 95]. Also see Beck and Chen [25] and Richert [258].

We can also consider the so-called L^2 discrepancy, T_n, defined as follows: let

$$R_n(t) = \frac{S_n([0, t), \omega)}{n} - t$$

and put

$$T_n(\omega) = \left(\int_0^1 R_n^2(t) dt \right)^{1/2}.$$

It is possible to generalize the definitions of D_n and T_n to the multi-dimensional case, though we omit the details. By appealing to numbers with bounded partial quotients, Davenport [73] constructed sequences in two dimensions with low L^2 discrepancy. Also see Proinov [250, 251, 252].

Another measure connected with sequences is called *dispersion*. Let $\omega = (x_1, x_2, \ldots)$ and define the dispersion

$$d_n(\omega) = \sup_{x \in [0,1]} \min_{1 \leqslant k \leqslant n} |x - x_k|,$$

essentially half the distance between the most widely separated points of the sequence $x_1, x_2, \ldots, x_n$. (Compare with the function $\delta_\theta(n)$ in Section 11.)

Niederreiter [221] considered the dispersion of the sequence $\{n\theta\}$. He showed that if θ has bounded partial quotients, then $d_n(\omega) = O(1/n)$. He also gave a more detailed estimate, showing that $d_n(\omega)$ is approximately $K(\theta)/4n$. Also see Drobot [93] and Larcher [311].

13. Connections with Ergodic Theory

Let θ be irrational, $\omega = (\theta, 2\theta, \ldots)$ and $S_n(I, \omega)$ be defined as in the previous section. Veech [293] developed connections between S_n and ergodic theory. We mention one result that is number-theoretic in nature. Let $x_n = S_n(I, \omega) \bmod 2$, and define

$$\mu_\theta(I) = \lim_{n \to \infty} \frac{1}{n} \sum_{1 \leqslant k \leqslant n} x_k,$$