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THE SUM OF THE CANTOR SET WITH ITSELF

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In a recent paper [2], Pavone gave an interesting geometric proof of the fact $C + C = [0, 2]$ where C denotes the Cantor ternary set. He also noted, as a consequence of his proof, that for any $k \in [0, 2]$ there exists either a finite or an uncountable number of pairs $(x, y) \in C \times C$ for which $x + y = k$. In the finite case, he also gives an unfortunately incorrect formula for the number of such pairs.

In this note we give a very simple proof of the fact that $C + C = [0, 2]$. From this proof it is also easy to count the number of representations of numbers in $[0, 2]$ as a sum of two elements of C and obtain a correct formula in the finite case. The proof given below that $C + C = [0, 2]$ is not new. It is, perhaps, the intended solution to an exercise in [1], and it is very similar to a "Quicky" proposed and solved by Shallit very recently in [3].

It is well known that $C = \{\sum 2\varepsilon_n/3^n : \varepsilon_n = 0 \text{ or } 1\}$. $C + C = [0, 2]$ is equivalent to $\frac{1}{2}C + \frac{1}{2}C = [0, 1]$. Also $\frac{1}{2}C = \{\sum \varepsilon_n/3^n : \varepsilon_n = 0 \text{ or } 1\}$ and hence

$$\begin{aligned} \tfrac{1}{2}C + \tfrac{1}{2}C &= \{\textstyle\sum (\varepsilon_n + \varepsilon_n')/3^n : \varepsilon_n = 0 \text{ or } 1 \quad \text{and} \quad \varepsilon_n' = 0 \text{ or } 1\} \\ &= \{\textstyle\sum a_n/3^n : a_n = 0, 1, 2\} = [0, 1] \end{aligned}$$

and the proof that $C + C = [0, 2]$ is complete.

Now we consider the number of representations of a number in $(0, 2)$ as a sum of two elements of C. Fix $k = 2h$ in $(0, 2)$ and let $h = \sum a_n/3^n$ be the unique infinite ternary expansion of h. Pavone claimed that: "...the equation $x + y = k$ has a finite or an uncountable number $S(k)$ of solutions in $C \times C$ according to whether the cardinality $c(k)$ of the set $\{n \in \mathbf{N}\setminus\{0\}; a_n = 1\}$ is finite or infinite respectively. In fact the exact formula is $S(k) = 1$ if $c(k) = 0$ or 1, and $S(k) = 3(2^{c(k)-2})$ otherwise." The statement concerning when $S(k)$ is finite or uncountable is correct, but the formula for $S(k)$, when finite, is not correct. It is not difficult to obtain the correct formula for $S(k)$, but different cases must be considered.

First consider the case where h has a unique ternary expansion, which is necessarily infinite. Then $S(k) = 2^{c(k)}$. To see, this, set $h = \sum a_n/3^n$ where $a_n = 0, 1,$ or 2 ($n = 1, 2, 3, \ldots$), $a_n \neq 0$ for infinitely many n and $a_n \neq 2$ for infinitely many n. We wish to count the number of representations $h = x + y$ where $x, y \in \frac{1}{2}C$; i.e., $x = \sum \varepsilon_n/3^n$, $y = \sum \varepsilon_n'/3^n$ and $\varepsilon_n, \varepsilon_n' = 0$ or 1. Now if $a_n = 0$, clearly $\varepsilon_n = \varepsilon_n' = 0$. Also if $a_n = 2$, $\varepsilon_n = \varepsilon_n' = 1$. However if $a_n = 1$, we can have $\varepsilon_n = 1$ and $\varepsilon_n' = 0$ or we can have $\varepsilon_n = 0$ and $\varepsilon_n' = 1$. Hence there are $2^{c(k)}$ choices for (x, y) (uncountable if $c(k)$ is infinite).

Next consider the case where h has two ternary expansions. Then they are necessarily of the form

$$h = .\, a_1 a_2 \ldots a_r 22 \ldots = .\, a_1 a_2 \ldots a_{r-1} b_r$$

where $a_1, a_2, \cdots, a_{r-1} = 0, 1, 2$, $a_r = 0$ or 1 and $b_r = a_r + 1$. Then using the ideas in the last paragraph and keeping in mind there are two counts (one for each representation of h), we have:

$$S(k) = \begin{cases} 3(2^{c(k)}) & \text{if} \quad a_r = 0 \,. \\ 3(2^{c(k)-1}) & \text{if} \quad a_r = 1 \,. \end{cases}$$

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