IV. Trepeau's ex ample

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Since K is invariant under rotation in the w variable:

$$\sup_K |\tilde{P}| \leqslant \sup_K |P|.$$

Set $Q(z_1, z_2) = P(0, z_1, z_2)$. Using (ii) one gets

$$|P(0, \zeta_1, \zeta_2)| = |Q(\zeta_1, \zeta_2)| \leqslant \sup_{\cup K_t} |Q| = \sup_K |\tilde{P}| \leqslant \sup_K |P|.$$

So (i) is established.

Remark. There is another approach to Lemma 3, which may better "explain" the situation, and that we just sketch. If $\varphi: \Delta \to \mathbf{C}^2$ is a holomorphic disk (φ continuous on $\bar{\Delta}$, holomorphic on Δ) and T is a continuous map from $\mathbf{R}/2\pi\mathbf{Z}$ into $[0, t_0]$ so that $\varphi(e^{i\theta}) \in K_{T(\theta)}$ $(\theta \in [0, 2\pi))$, then $\varphi(0) \in \widehat{\cup K_t}$. One sees that $(0, \varphi(0)) \in \hat{K}$ by considering holomorphic disks $(Q, \varphi): \Delta \to \mathbf{C} \times \mathbf{C}^2$, with $Q(0) = 0$ and $|Q(e^{i\theta})| \simeq T(\theta)$. Carrying this out in general may require the use of the fundamental theorem by Poletsky [6], which says that, in an appropriate sense, polynomial hulls are always explained by holomorphic disks.

IV. Trepeau's example

Here we describe a class of examples. Let χ be a smooth real valued function defined on $[0, 1]$, constant in no neighborhood of 0, and so that $\chi(0) = 0$, $|\chi| < 1$. In one of the versions of Trepreau's original example $\chi(t) = t$. Let $\mathcal{M}$ be the generic 4-dimensional manifold in $\mathbf{C}^3$, given by:

$$\mathcal{M} = \{(w, z_1, z_2) \in \mathbf{C}^3, |w| < 1, z_1 = s_1 + i\chi(|w|^2)s_2,$$
$$z_2 = s_2 - i\chi(|w|^2)s_1;\ (s_1, s_2) \in \mathbf{R}^2\}.$$

Notice that on $\mathcal{M}$, $z_1^2 + z_2^2$ is a real valued function, (on $\mathcal{M}$, $z_1^2 + z_2^2 \geqslant 0$), hence:

(*) *Any function which depends only on* $(z_1^2 + z_2^2)$ *is a CR function on* $\mathcal{M}$.

This already gives example of *CR* functions which cannot be holomorphically extended to any wedge. The existence of such functions is related to the fact that $\mathcal{M}$ is not "minimal" (in the sense of Tumanov), it contains $\mathbf{C} \times \{0\} \times \{0\}$ as a (nongeneric) *CR* manifold of same *CR* dimension (see [9], [2]).

Before going any further, we wish to rewrite the definition of $\mathscr{M}$ in the spirit of II and III. With the notations used in II:

$$\mathscr{M} = \{(w, z_1, z_2) \in \mathbf{C}^3, |w| < 1, (z_1, z_2) \in \mathbf{R}^2_{\chi(|w|^2)}\} .$$

PROPOSITION 1. *There are smooth CR functions on* $\mathscr{M}$ *which in no neighborhood of* 0 *can be decomposed into the sum of boundary values of functions holomorphic in wedges (with edge* $\mathscr{M}$*).*

There is some ambiguity in the statement since it is not made precise in which sense boundary values are taken. To keep things at the most elementary level we will treat in detail the case of *continuous boundary values.* See the remark below, and V, for the case of more general boundary values. (Although I suspect that one can prove, as a general fact, that if every *smooth CR* function is decomposable, then the decomposition can be done with functions continuous (and even smooth) up to the edge).

Proof. We can assume that in any neighborhood of 0, χ takes some strictly positive values (permuting the variables z_1 and z_2, if needed). Let $\mathscr{W}$ be an arbitrary wedge, with edge $\mathscr{M}_0$ the intersection of $\mathscr{M}$ with some neighborhood of 0.

The reader willing to read V will see *that every CR function on* $\mathscr{M}_0$ *which has a holomorphic extension to some wedge with edge* $\mathscr{M}_0$ *is analytic in some neighborhood of* 0.

The reader unwilling to read V, and willing to use only the simple techniques used in II and III will have to use the "subclaim".

"SUBCLAIM". *Let* f *be a continuous CR function on* $\mathscr{M}_0$, *which has a holomorphic extension to* $\mathscr{W}$. *Then there exists* $\varepsilon > 0$, *and* V *the intersection of a neighborhood of* 0 *in* $\mathbf{C}^2$ *with a neighborhood of* $\Sigma_\varepsilon - \{0\}$ *so that the function* $(x_1, x_2) \mapsto f(0, x_1, x_2)$ *has a continuous extension to* $\bar{V}$, *holomorphic on the interior of* V.

Proof of the subclaim. After shrinking of $\mathscr{W}$ and $\mathscr{M}_0$, the Baouendi-Treves approximation formula ([3], [8] II.2) shows that f is the uniform limit on $\bar{\mathscr{W}}$ of a sequence of polynomials (P_j).

For Γ an open cone in $\mathbf{C}^2$ and $\rho > 0$, and $w \in \mathbf{C}\,|\,w| < \rho$, we consider $K_{|w|}$ the closure in $\mathbf{C}^2$ of the wedge $W(\mathbf{R}^2_{\chi(|w|^2)} \cap B(0, \rho), \Gamma, e)$, (with edge in $\mathbf{R}^2_{\chi(|w|^2)}$). One can choose Γ and ρ so that for every $w \in \mathbf{C}, |w| < \rho$, we have $\{w\} \times K_{|w|} \subset \bar{\mathscr{W}}$. We apply Lemma 3 to these sets $K_{|w|}$ and to the set $K = \cup(\{w\} \times K_{|w|})$.

And Lemma 1 then gives that the polynomial hull of $\bar{\mathscr{W}}$ contains $\{0\} \times \bar{V}$, with V as in the claim. The sequence of approximating polynomials converges uniformly on $\bar{V}$ to a function which provides the desired extension. The subclaim is thus proved.

Now we finish the proof of the proposition. Let φ be a function on $\mathbf{R}^+$ which is not analytic at 0, or, for the reader willing to use only the "subclaim", so that the function $(x_1, x_2) \mapsto \varphi(x_1^2 + x_2^2)$ does not have a continuous extension to $\bar{V}$, holomorphic on the interior of V, for any V intersection of a neighborhood of 0 in $\mathbf{C}^2$ with a neighborhood of $\Sigma_\varepsilon - \{0\}$. Any smooth function φ nonidentically zero but vanishing on open intervals in any neighborhood of 0 has this property. As pointed out (*), the function $(w, z_1, z_2) \mapsto \varphi(z_1^2 + z_2^2)$ in a *CR* function on $\mathscr{M}$. It follows from V or the subclaim that it is a nondecomposable one. It cannot be written as the sum of continuous boundary values of holomorphic function on wedges.

Remark. There are some few technical details (such as precising the shape of V) to be dealt with, to adapt the approach that we have just used to the case of boundary values distributions. In this setting the Baouendi Treves approximation still gives approximation by polynomials (on wedges, with locally uniform convergence, and with uniformly controlled polynomial growth when approaching the edge). Also, one can still speak about the restriction of a *CR* distribution on $\mathscr{M}$ to $\{0\} \times \mathbf{R}^2(f(0, x_1, x_2))$, (this is a basic fact used to define mini F.B.I, see [8] Corollary I.4.1.).

But it seems pointless to go into this. Indeed this kind of difficulties merely disappear when using the results explained in the next paragraph.

V. More

1) In Lemma 1, the right conclusion is in fact that 0 belongs to the interior of the polynomial hull of $\bar{W}$. Applying Lemma 1, with trivial homogeneity considerations, and replacing $\mathbf{R}^2_\varepsilon$, by $\mathbf{R}^2$, it reduces to the following proposition.

PROPOSITION 2. *Let f be a function defined on some neighborhood of 0 in $\mathbf{R}^2$. Assume that near 0, f extends holomorphically to a conic neighborhood of $\mathbf{R}^2 - \{0\}$, and also to a wedge with edge $\mathbf{R}^2$. Then f is analytic at 0.*

By conic neighborhood, we mean a cone which is a neighborhood of $\mathbf{R}^2 - \{0\}$ in $\mathbf{R}^2$. We did not make precise whether f is continuous, but