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From Lemma 3.5, it follows then that

$$B \subseteq \bigcup_{\varepsilon_1, \ldots, \varepsilon_{35} \in \{0, 1\}} \left[\left(\sum_{j=1}^{n} \varepsilon_j z^j \right) + \left(\frac{1}{2} + O(|\delta|) \right) z^n B \right]$$

so for sufficiently small δ, we may apply Lemma 3.1 to deduce $z \in \bar{W}$. □

We now combine all the results of this section.

THEOREM 3.1. *There is an open neighborhood of* $\{z : |z| = 1, z \neq 1\}$ *contained in* $\bar{W}$.

Proof. Apply Propositions 3.2 and 3.3. □

COROLLARY 3.1. *If* $z \in (-1, -1 + \delta)$ *for sufficiently small* δ *then* z *is a multiple zero of some* 0, 1 *power series.*

Proof. By Theorem 3.1, if δ is small enough we can pick 0, 1 power series f_n and zeros z_n of f_n such that $z_n \notin \mathbf{R}$ and $z_n \to z$ as $n \to \infty$. By taking a subsequence we may assume that the coefficient of z^k in f_n is eventually constant for large n, for each k. By a Rouché's Theorem argument, the pairs of zeros $\{z_n, \bar{z}_n\}$ of f_n must converge to (at least) a double zero at z of $\lim_{n \to \infty} f_n$. □

4. $\bar{W}$ IS CONNECTED

Since W is countable, we cannot hope to prove W is connected. We prove instead that $\bar{W}$ is connected. First we need some topological lemmas.

Give $\{0, 1\}$ the discrete topology and $\{0, 1\}^\omega$ the product topology, as usual. If $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)$ is a finite vector of 0's and 1's, let S_υ be the set of sequences in $\{0, 1\}^\omega$ which start with υ. The following lemma is the key ingredient in the connectivity proof.

LEMMA 4.1. *Let* Y *be a topological space. Suppose* $f : \{0, 1\}^\omega \to Y$ *is a continuous map such that*

$$f(S_{\upsilon 0}) \cap f(S_{\upsilon 1}) \neq \emptyset \tag{4.1}$$

for all $\upsilon \in \{0, 1\}^n$, *and all* $n \geqslant 0$. *(Here* $\upsilon 0$ *denotes the vector* υ *with* 0 *appended, etc.) Then the image of* f *is path connected.*

Proof. Let $w(0) = f(x'_0)$ and $w(1) = f(x_1)$ be elements of the image we wish to connect by a path. Find $x_{1/2}, x'_{1/2} \in \{0, 1\}^\omega$ such that $x'_0, x_{1/2}$ have the same first coordinate, and $x'_{1/2}, x_1$ have the same first coordinate and $f(x_{1/2}) = f(x'_{1/2})$. (If x'_0, x_1 have the same first coordinate, take $x_{1/2} = x'_{1/2} = x'_0$; otherwise apply the hypothesis (4.1) with υ as the empty vector.) Let $w(1/2)$ be this common value.

Next find $x_{1/4}, x'_{1/4} \in \{0, 1\}^\omega$, using the same argument, such that $x'_0, x_{1/4}$ agree in the first two coordinates, $x'_{1/4}, x_{1/2}$ agree in the first two coordinates, and $f(x_{1/4}) = f(x'_{1/4})$. Let $w(1/4)$ be this common value. Do the analogous thing at 3/4.

By induction, we may continue to define $x_{d/2^n}, x'_{d/2^n}, w(d/2^n)$ at all dyadic rationals $d/2^n$ in $[0, 1]$, such that $x'_{d/2^n}$ and $x_{(d+1)/2^n}$ agree in the first n coordinates and

$$w(d/2^n) = f(x_{d/2^n}) = f(x'_{d/2^n}) .$$

By induction, we see that all the x'_q with $q \in [d/2^n, (d+1)/2^n)$ agree in the first n coordinates. Hence for

$$r = \sum_{i=1}^{\infty} \varepsilon_i 2^{-i} \in [0, 1], \qquad \varepsilon_i \in \{0, 1\}$$

not a dyadic rational, we may define

$$x_r = x'_r = \lim_{n \to \infty} x'_{\sigma(n)} \quad \text{where} \quad \sigma(n) = \sum_{i=1}^{n} \varepsilon_i 2^{-i},$$

and $w(r) = f(x_r)$. Then w maps $[d/2^n, (d+1)/2^n]$ into $f(S_\upsilon)$ where $\upsilon \in \{0, 1\}^n$ is the first n coordinates of x'_r, $r \in [d/2^n, (d+1)/2^n)$ and of $x_{(d+1)/2^n}$.

We now show that w is continuous at $r \in [0, 1]$. Let U be an open set of Y containing $w(r)$. Then $f^{-1}(U)$ contains S_υ and $S_{\upsilon'}$ for some finite substrings υ, υ' of x_r, x'_r respectively, by continuity of f. By the last sentence of the previous paragraph it follows that

$$w^{-1}(U) \supseteq w^{-1}(f(S_\upsilon) \cup f(S_{\upsilon'}))$$

will contain a neighborhood of r.

Thus $w: [0, 1] \to \text{image}(f)$ is a continuous path, and image (f) is path connected. $\square$

Let M be a topological space. Give M^n the product topology and let the symmetric group S_n act on M^n by permuting the coordinates. The space

M^n/S_n, which parameterizes n-element multisets, can be given the quotient topology.

LEMMA 4.2. *If* $A \subseteq M^n/S_n$ *is connected, and the multiset* $\{P, P, \ldots, P\}$ *is in* A *for some* $P \in M$, *then the subset* $B \subseteq M$ *of all coordinates of points in* A *is connected.*

Proof. Suppose not. Then there are open sets $U, V \subseteq M$ such that $U \cap B$ and $V \cap B$ are disjoint nonempty sets with union B. Without loss of generality, $P \in U$. Let

$$U' = U \times U \times \cdots \times U,$$

$$\begin{aligned} V' = {} & (V \times M \times M \times \cdots \times M) \\ & \cup (M \times V \times M \times \cdots \times M) \\ & \qquad \vdots \\ & \cup (M \times M \times M \times \cdots \times V). \end{aligned}$$

Then U', V' are open sets in M^n which are stable under S_n, so they project to open sets U'', V'' in M^n/S_n. Also $A \subseteq U'' \cup V''$ since a point in A must have all coordinates in U, or else at least one coordinate in $B \backslash U \subseteq V$. Furthermore $P \in U'' \cap A$, and $V'' \cap A$ is nonempty also, since at least one point of A has a coordinate in V, since $V \cap B \neq \emptyset$. Finally $U'' \cap V'' \cap A = \emptyset$, since it is not possible for a point of A to have all coordinates in U, yet have some coordinate in V. This contradicts the connectedness of A. □

THEOREM 4.1. $\bar{W}$ *is connected.*

Proof. First we show that for $\delta \in (0, 1)$,

$$\bar{W}_\delta = (\bar{W} \cap \{z : |z| \leqslant 1\}) \cup \{z : 1 - \delta \leqslant |z| \leqslant 1\}$$

is connected. The idea is to apply Lemma 4.1 to the function f which assigns to $(\varepsilon_1, \varepsilon_2, \ldots)$ the set of zeros of

$$1 + \varepsilon_1 z + \varepsilon_2 z^2 + \cdots$$

inside $\{z : |z| < 1 - \delta\}$. To make a continuous map of this requires some manipulation.

By Jensen's theorem, as was shown in Section 2, there is an upper bound n on the number of zeros that a power series with 0, 1 coefficients can have inside $\{z : |z| < 1 - \delta\}$. Let M be $\{z : |z| \leqslant 1\}$ with the annulus

$\{z: 1 - \delta \leqslant |z| \leqslant 1\}$ shrunk to a point P. (Therefore M is topologically a sphere.) To each power series $1 + \sum_{i=1}^{\infty} \varepsilon_i z^i$, $\varepsilon_i \in \{0, 1\}$, we assign the set of zeros inside $\{z: |z| < 1 - \delta\}$, (counted with multiplicities) and throw in extra copies of the point P as necessary to bring the total number of points to n. Since the order of these n elements of M is unspecified, we obtain a point of M^n/S_n. Let $f((\varepsilon_1, \varepsilon_2, \ldots))$ be this point.

We claim that this map

$$f: \{0, 1\}^{\omega} \to M^n/S_n$$

is continuous. This follows easily from Rouché's theorem; if two power series agree in the first m coordinates for m sufficiently large then their zeros inside $\{z: |z| < 1 - \delta\}$ will be within ε. Some may escape or enter the disk, but this is not a problem, since in the topology on M, P is close to all points z with $|z|$ sufficiently near $1 - \delta$.

We next check condition (4.1) of Lemma 4.1. This is easily done using the following trick: given

$$v = (v_1, v_2, \ldots, v_n) \in \{0, 1\}^n,$$

let $w = (v_1, v_2, \ldots, v_n, 1, v_1, v_2, \ldots, v_n)$. Then $v \in S_{v0}$, $w \in S_{v1}$, and $f(v) = f(w)$ (we extend v, w to infinite vectors by appending 0's), since

$$1 + v_1 z + v_2 z^2 + \cdots + v_n z^n$$

and

$$1 + v_1 z + v_2 z^2 + \cdots + v_n z^n + z^{n+1} + v_1 z^{n+2} + \cdots + v_n z^{2n+1}$$
$$= (1 + v_1 z + v_2 z^2 + \cdots + v_n z^n)(1 + z^{n+1})$$

have the same zeros inside $\{z: |z| < 1 - \delta\}$. Therefore we may apply Lemma 4.1 and deduce that the image of f is path connected.

Since $f((0, 0, \ldots)) = (P, P, P, \ldots, P)$, we may apply Lemma 4.2 with $A = \text{image}(f)$ to deduce that $\overline{W}_\delta$ with the annulus $\{z: 1 - \delta \leqslant |z| \leqslant 1\}$ shrunk to a point P is a connected subset of M. This is equivalent to the connectivity of $\overline{W}_\delta$.

Since $\overline{W} \cap \{z: |z| \leqslant 1\}$ is the decreasing intersection of the compact connected sets $\overline{W}_{1/m}$, it too is connected. So is its image under $z \mapsto 1/z$. Finally, $\overline{W}$ is the union of these two sets, which meet on the unit circle, so $\overline{W}$ is connected as well. □