§1. Introduction

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HURWITZ QUATERNIONIC INTEGERS AND SEIFERT FORMS

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Dedicated to the memory of late Prof. K. G. Ramanathan

§ 1. INTRODUCTION

The aim of this paper is to answer a question which arose from the work of Kervaire [K] on Seifert forms.

A *Seifert form* B on a finitely generated free $\mathbf{Z}$-module L, is a bilinear form

$$B: L \times L \to \mathbf{Z}$$

such that $B + B'$ is unimodular, i.e. $\det(B + B') = \pm 1$, where B' denotes the transpose of B. Such forms occur in knot theory. The Seifert form associated with the fibres of an odd dimensional fibred knot is unimodular. Motivated by this, M. Kervaire considers in [K] the following question:

1.1. QUESTION. Let S be a unimodular symmetric bilinear form on a finitely generated free $\mathbf{Z}$-module L. Does there exist a unimodular form

$$B: L \times L \to \mathbf{Z} ,$$

such that $S = B + B'$?

If S admits such a decomposition, then obviously B is not symmetric and S is even. If S is indefinite, the answer to the above question is easily shown to be in the affirmative if and only if the rank of L exceeds 2 ([K], p. 176). To answer the question in the positive definite case, Kervaire introduces the notion of a perfect isometry.

1.2. *Definition.* Let R be a commutative ring and M a finitely generated R-module. An R-linear isomorphism τ of M is called *perfect* if $1 - \tau$ is invertible.

He proves:

1.3. Proposition. *A unimodular symmetric bilinear form* S *admits a decomposition* $S = B + B'$ *with* B *unimodular if and only if* S *has a perfect isometry.*

Thus, Question 1.1 reduces to the following.

1.4. Question. Given a unimodular symmetric bilinear form S, does there exist a perfect isometry of S?

Note that if S is positive definite and even, then the rank of S is a multiple of 8. M. Kervaire gives a complete answer to Question 1.4, for positive definite forms of rank less than or equal to 24. For forms of arbitrary rank, he proves the following partial result, using the theory of the associated root systems.

Let $\mathrm{R} = \{x \in L \mid S(x, x) = 2\}$. Suppose that R is a root system in $\mathbf{R}^n$ of rank n ($= \operatorname{rank} L$). Then the irreducible components of R are of type A, D, or E; and we have:

1.5. Theorem ([K], Cor. 3, Prop. 4).

(a) *If* R *has an irreducible component of type* A_{2k-1}, E_7 *or* D_{k+4}, *with* $k \geqslant 1$, *then there does not exist any perfect isometry of* (L, S).

(b) *If* $\mathrm{R} = \perp_{1 \leqslant i \leqslant p} \mathrm{A}_{2k_i} \perp q\mathrm{E}_6 \perp r\mathrm{E}_8$, *then there exists a perfect isometry of* L, *inducing a perfect isomorphism of the abelian group* $\mathbf{Z}\mathrm{R}^{\#}/\mathbf{Z}\mathrm{R}$, *which corresponds to multiplication by* -1, *where* $\mathbf{Z}\mathrm{R}^{\#}$ *denotes the dual of the lattice* $\mathbf{Z}\mathrm{R}$.

Note that the case of R having an irreducible component of type D_4 is not covered by this theorem. In this paper we give an analogue of (b) for this case. In fact, we first consider the case in which R is of type $n\mathrm{D}_4$. In this case, we show (Th. 5.2) that (L, S) admits a perfect isometry if and only if the isometry class of (L, S) contains a symmetric bilinear space (L', S') of some hermitian space over the Hurwitz quaternionic integers. The analogue of Proposition 1.5 follows from this immediately (Theorem 5.3). In the final section we also give some examples.

§2. The root system D_4 and the Hurwitz quaternionic integers

The fact that the root lattice $\mathbf{Z}\mathrm{D}_4$ can be identified with the lattice of Hurwitz quaternionic integers was long recognized: see for instance ([C-S]). However we give here a direct proof of this fact and recall some arithmetical facts about these quaternionic integers, which are needed in the sequel.