2.2. Problems

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cients. The proof of Theorem 2 is based on a trace formula. We do not give here any more details. Good expositions can be found in [9] and [39].

2.2. Problems

i) Since for fixed k the dimension of $J_{k,m}$ grows linearly in m, the map ρ_m defined by (3) for $m \gg_k 0$ cannot be surjective. Is there any simple or nice description of the image of ρ_m or $(im\,\rho_m \mid S_k(\Gamma_2))^{\perp}$? Let us mention here that one can express the Fourier-Jacobi coefficients of Poincaré series of exponential type on Γ_2 which generate $S_k(\Gamma_2)$, as certain infinite linear combinations of Poincaré series on Γ_1^J [22]. Taking scalar products one obtains a characterization of $(im\,\rho_m \mid S_k(\Gamma_2))^{\perp}$ as the kernel of certain infinite systems of linear equations. This description, however, does not seem to be very illuminating (for example, it does not imply in any obvious way that ρ_1 is surjective).

ii) A skew-holomorphic Jacobi form of weight $k \in \mathbf{Z}$ and index $m \in \mathbf{N}_0$ on Γ_1^J as introduced by Skoruppa is a complex-valued C^∞-function $\phi(\tau, z)$ ($\tau \in \mathcal{H}, z \in \mathbf{C}$) satisfying the following properties: 1) ϕ is holomorphic in z and is annihilated by the heat operator $8\pi i m \partial/\partial\tau - \partial^2/\partial z^2$; 2) ϕ satisfies the same transformation formula under Γ_1^J as a holomorphic Jacobi form of weight k and index m (cf. § 1.2) except that the factor $(c\tau + d)^k$ has to be replaced by $(c\bar{\tau} + d)^{k-1} \mid c\tau + d \mid$; 3) ϕ has a Fourier expansion of type

$$\phi(\tau, z) = \sum_{n, r \in \mathbf{Z}, r^2 \geqslant 4mn} c(n, r) \exp\left(-\pi \frac{r^2 - 4mn}{m} v\right) q^n \zeta^r \qquad (v = \mathrm{Im}(\tau)) .$$

Note that a skew-holomorphic Jacobi form of even weight and index 1 is identically zero as is easily seen.

Despite of the importance of skew-holomorphic Jacobi forms as demonstrated in [34, 36] it is not quite clear so far how they are related to Siegel modular forms. One difficulty, for example, is that if one starts with a real-analytic Siegel modular form of genus 2, the coefficients of the partial Fourier expansion of $F(Z)$ w.r.t. $e^{2\pi i \tau'}$ $\left(\text{where as usual } Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right)$ not only depend on τ and z but also on $\mathrm{Im}(\tau')$, and it is a priori not obvious how to get rid of the latter variable and to produce "true" Jacobi forms.

Let k be an odd integer and denote by $M_{1/2, k-1/2}(\Gamma_2)$ the space of Siegel-Maass wave forms "of type $(1/2, k - 1/2)$" as defined in [26], i.e. the space of real-analytic functions $F: \mathcal{H}_2 \to \mathbf{C}$ which satisfy

$$F(M\langle Z\rangle) = \det(C\bar{Z} + D)^{k-1} \mid \det(CZ + D) \mid F(Z)$$

for all $M = \begin{pmatrix} \cdot & \cdot \\ C & D \end{pmatrix} \in \Gamma_2$ and which are annihilated by the matrix differential operator

$$\Omega_{1/2,k-1/2} := (Z-\bar{Z})\left((Z-\bar{Z})\frac{\partial}{\partial Z}\right)'\frac{\partial}{\partial \bar{Z}} + \frac{1}{2}(Z-\bar{Z})\frac{\partial}{\partial \bar{Z}}$$
$$- \left(k-\frac{1}{2}\right)(Z-\bar{Z})\frac{\partial}{\partial Z}$$

$$\text{where} \quad \frac{\partial}{\partial Z} = \begin{pmatrix} \dfrac{\partial}{\partial \tau} & \dfrac{1}{2}\dfrac{\partial}{\partial z} \\ \dfrac{1}{2}\dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial \tau'} \end{pmatrix}$$

and $\dfrac{\partial}{\partial \bar{Z}}$ is defined analogously (the notation "of type $(1/2, k-1/2)$" comes from the fact that the factor of automorphy of F can be written as $\det(C\bar{Z}+D)^{k-1/2}\det(CZ+D)^{1/2}$ with appropriate choice of the square root).

Using certain invariance properties of $\Omega_{1/2,k-1/2}$ under the action of $Sp_2(\mathbf{R})$ one can define Hecke operators $T_n (n \in \mathbf{N})$ on $M_{1/2,k-1/2}(\Gamma_2)$ in the usual way. Let

$$E^{(2)}_{1/2,k-1/2}(Z) := \sum_{(C,D)} \det(CZ+D)^{-k+1}\,|\det((CZ+D))|^{-1} \qquad (k>3)$$

be the Maass-Siegel-Eisenstein series in $M_{1/2,k-1/2}(\Gamma_2)$ ([26; 27, § 18]; summation over all pairs (C, D) of relatively prime symmetric (2, 2)-matrices inequivalent under left-multiplication by $GL_2(\mathbf{Z})$). Then the following can be shown:

1) The function $E^{(2)}_{1/2,k-1/2}$ is a Hecke eigenform whose spinor zeta function (defined in the same way as above) is equal to $\zeta(s-k+1)\zeta(s-k+2)L_{E_{2k-2}}(s)$ where E_{2k-2} is the normalized Eisenstein series of weight $2k-2$ on Γ_1 (this implies that $E^{(2)}_{1/2,k-1/2}$ for all primes p is annihilated by the Hecke operator $\mathscr{E}_p$ defined analogously as in (5));

2) if $e_{1/2,k-1/2;m}(\tau, z, \mathrm{Im}(\tau'))$ is the m-th Fourier-Jacobi coefficient of $E^{(2)}_{1/2,k-1/2}$ and if for $m>0$ one carries out a similar limit process as in [19, § 2, Remark ii) after the proof of Thm. 1], i.e. essentially replaces $\mathrm{Im}(\tau')$ by $(\mathrm{Im}(z))^2/\mathrm{Im}(\tau)+\delta$ and lets $\delta \to \infty$, then one obtains a skew-holomorphic Eisenstein series of weight k and index m (in fact, finite linear combinations of such Eisenstein series if m is not squarefree).

The following questions therefore are suggestive:

1) if one starts with an arbitrary $F \in M_{1/2, k-1/2}(\Gamma_2)$, does the above limit process produce skew-holomorphic Jacobi forms of weight k?

2) define $M^*_{1/2, k-1/2}(\Gamma_2)$ as the subspace of $M_{1/2, k-1/2}(\Gamma_2)$ consisting of the intersection of the kernels of the operators $\mathscr{C}_p$ for all primes p. Does there exist a natural map V from skew-holomorphic Jacobi forms of weight k and index 1 to $M^*_{1/2, k-1/2}(\Gamma_2)$ similar as in the case of holomorphic Jacobi forms?

Recently, N.-P. Skoruppa [36] has developed a theory of theta lifts from skew-holomorphic Jacobi forms to automorphic forms on Sp_2. It would be interesting to investigate if his lifts would provide (at least partial) answers to the above questions.

iii) So far a generalization of the Maass space to higher genus $n > 2$ has not been given; in fact, in the general case it does not seem to be quite clear what one has to look for, except that (the cuspidal part) of a "Maass space" eventually should be generated by Hecke eigenforms which do not satisfy a generalized Ramanujan-Petersson conjecture. Note that there is a partial negative result by Ziegler [40, 4.2. Thm.] who showed by means of specific examples that for $n \geqslant 33$ the map which sends a Siegel modular form of weight 16 on $\Gamma_n := \mathrm{Sp}_n(\mathbf{Z})$ to its first Fourier-Jacobi coefficient is not surjective.

On the other hand, there are very interesting numerical calculations for $n = 3$ due to Miyawaki [30] which suggest that a Siegel-Hecke eigenform F of even integral weight k on Γ_3 could be constructed from a pair (f, g) of elliptic Hecke eigenforms of weights (k_1, k_2) equal to $(k, 2k-4)$ or $(k-2, 2k-2)$ such that the (formal) spinor zeta function of F should be equal to $L_f(s - k_2/2) L_f(s - k_2/2 + 1) L_{f \otimes g}(s)$ where $L_{f \otimes g}(s)$ essentially is the Rankin convolution of f and g ([*loc. cit.*, §4]; note that for $n > 2$ the analytic continuation of the spinor zeta function of a holomorphic Hecke eigenform on Γ_n is not known).

§3. Spinor zeta functions

3.1. Results

Although the Maass space $S^*_k(\Gamma_2)$ as discussed in the previous section is an important subspace of $S_k(\Gamma_2)$ in its own right, one quickly realizes that the "true" Siegel cusp forms on Γ_2 should lie in the orthogonal complement of $S^*_k(\Gamma_2)$ (cf. Theorem 2 in §2 and its discussion). Is is therefore even more