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by Nils-Peter SKORUPPA

ABSTRACT. A short and simple proof for Zimmert's lower bounds for regulators of number fields is presented.

1. INTRODUCTION. Let K be an algebraic number field with r_1 real and $2r_2$ complex embeddings, let R denote its regulator and w its number of roots of unity. The purpose of this note is to present a surprisingly short proof of the following theorem.

THEOREM. *For any real number* $s > 1$ *one has*

$$\frac{R}{w} \geqslant \frac{s(s-1)}{e} \exp\left(-\frac{s}{s-1}\right) \gamma(s) \exp\left(-s\frac{\gamma'}{\gamma}(s)\right),$$

Here $\gamma(s) = 2^{-r_1}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}$, *where* $\Gamma(s)$ *denotes the gamma function, and* $\gamma'(s)$ *is the derivative.*

If we let $s = 4/3$ then we obtain

$$\frac{R}{w} \geqslant 0.00299 \cdot \exp(0.48 r_1 + 0.06 r_2) .$$

From this one deduces that regulators of number fields are bounded from below by an absolute constant and grow exponentially in the degree of K.

This result and an estimate similar to the one of the above theorem was first stated and proved by Zimmert [Z, Satz 3 and Korollar]. He proved the sharper estimate

$$\frac{R}{w} \geqslant \frac{s(2s-1)}{2e} \exp\left(-\frac{2s}{s-1}\right) \gamma(2s) \exp\left(-2s\frac{\gamma'}{\gamma}(s)\right)$$

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(to compare this inequality to the inequality as originally stated by Zimmert one has to apply the gamma duplication formula). He chose $s = 2$ to obtain

$$\frac{R}{w} \geqslant 0.02 \cdot \exp(0.46 r_1 + 0.1 r_2) \ .$$

Zimmert deduced his regulator bounds by an ingenious, but quite involved, investigation of certain analytic properties of the partial Dedekind zeta function associated to the class of principal ideals of K.

In this note we show that it is possible to deduce the above theorem by a simple estimate from a certain, almost obvious, monotonicity property of Hecke's theta function associated to the maximal order of K (see below). Moreover, we indicate below how this method of proof can be refined to yield exactly Zimmert's bounds. The technique of estimating which we apply is a sort of simple variation of a method which is developped in [F-S] to obtain lower bounds for L^p-norms of a certain class of functions. It was found during a careful analysis of Zimmert's method and reflects, though it looks much easier, still very much the spirit of Zimmert's original proof.

2. PROOF. Let $|\cdot|_j$ for $1 \leqslant j \leqslant r := r_1 + r_2$ denote the archimedean absolute values of K, let G denote the r-fold direct product of the multiplicative group of the positive reals $\mathbf{R}_+$, and let V denote the image in G of the units of K under the map

$$\eta \mapsto (\ldots, |\eta|_j^{n_j}, \ldots) \ ,$$

where n_j equals 1 or 2 accordingly as $|\cdot|_j$ is real or complex. Denote by δ the group homomorphism

$$\delta : G \to \mathbf{R}_+ \ , \quad \delta((\ldots, x_j, \ldots)) = x_1 \cdots x_r \ .$$

Its kernel contains V, and by Dirichlet's unit theorem $\ker \delta / V$ is compact. We can thus fix a Haar measure μ on G/V by requiring

$$\int_{G/V} g \circ \delta \, d\mu = \frac{R}{w} \int_0^\infty g(t) \frac{dt}{t}$$

for any integrable function on $\mathbf{R}_+$. Let

$$Z(s) := \gamma(s) \sum_{\alpha \in \mathfrak{R}} |\mathrm{N}_{K/\mathbf{Q}} \alpha|^{-s} \ ,$$

where $\mathfrak{R}$ is a set of representatives for the non-zero elements of $\mathfrak{O}$, the ring of integers in K, modulo units. According to Hecke [H] (and according to the choice of μ) one has, for $\mathrm{Re}(s) > 1$, the integral representation

$$Z(s) = \int_{G/V} \Theta \delta^s d\mu .$$

Here Θ is a smooth, non-negative and V-invariant function on G, which is given by

$$\Theta(x) = \sum_{\substack{\alpha \in \mathfrak{D} \\ \alpha \neq 0}} \exp\left(- \sum_{j=1}^{r} |\alpha|_j^2 x_j^{2/n_j} \right) .$$

The main observation for the proof of the theorem is the

LEMMA. *The function* $(1 + \Theta)\delta$ *is increasing in each argument.*

Proof. This follows from Hecke's theta formula [H, p. 165-166]

$$(1 + \Theta(x))\delta(x) = \frac{\pi^{\frac{n}{2}} 2^{r_2}}{\sqrt{|d|}} \sum_{\alpha \in \mathfrak{D}^{-1}} \exp\left(- \pi^2 \sum_{j=1}^{r} n_j^2 |\alpha|_j^2 x_j^{-2/n_j} \right) ,$$

i.e. by applying Poisson summation to the series defining $\Theta(x)$ (here $\mathfrak{D}$ and d denote the different and discriminant of K). □

We can now give the

Proof of the theorem. For $a \in \mathbf{R}_+$ set

$$I(a) := \int_{G/V} (1 + \Theta(x))\delta(x) w((a\delta(x))^{s-1}) d\mu(x) ,$$

where we use

$$w(t) = t \max(0, \log(1/t)) ,$$

and where $s > 1$ as in the theorem.

For any $\varepsilon > 0$, one has $w(t) = O(t^{\varepsilon})$ and $|w(t+h) - w(t)| / |h| \leqslant |w'(t)| = O(t^{-\varepsilon})$ as $t \to 0$. Thus, using the convergence of the integral representation of $Z(s)$ for $s > 1$, we deduce that the integral defining $I(a)$ is finite, and, on applying Lebesgue's theorem, that $I(a)$ is differentiable and its derivative is obtained by differentiating under the integral sign. Here we agree to use $w'(1)$ for the derivative on the right, i.e. $w'(1) = 0$.

On replacing x by $x/a^{1/r}$ in the integral defining $I(a)$ we deduce from the lemma that $I(a)$ is decreasing. Hence $I'(a) \leqslant 0$, from which we obtain, writing $\sigma = s - 1$,

$$-\frac{d}{da}\int_{G/V} \delta w((a\delta)^{\sigma})d\mu \geqslant \frac{d}{da}\int_{G/V} \Theta\delta w((a\delta)^{\sigma})d\mu$$

$$= \sigma a^{\sigma-1}\int_{G/V} \Theta\delta^{s}w'((a\delta)^{\sigma})d\mu \geqslant -\sigma a^{\sigma-1}\int_{G/V} \Theta\delta^{s}(1+\log(a\delta)^{\sigma})d\mu$$

$$= -\sigma^{2}a^{\sigma-1}Z(s)\left(\frac{1}{\sigma} + \log a + \frac{Z'}{Z}(s)\right).$$

By the choice of μ the left-hand side equals $R\sigma/(ws^{2}a^{2})$. Multiplying the above inequality by $s^{2}a^{2}/\sigma$ and then maximizing the right hand side, i.e. choosing

$$\frac{1}{\sigma} + \log a + \frac{Z'}{Z}(s) = -\frac{1}{s},$$

we find

$$\text{(1)} \qquad \frac{R}{w} \geqslant \frac{s(s-1)}{e}\exp\left(-\frac{s}{s-1}\right)Z(s)\exp\left(-s\frac{Z'}{Z}(s)\right).$$

Finally, $Z(s) \geqslant \gamma(s)$, since the Dirichlet series $D(s)$ in the definition of $Z(s)$ satisfies $D(s) > 1$, and $\frac{Z'}{Z}(s) \leqslant \frac{\gamma'}{\gamma}(s)$, since $D'(s) < 0$. Thus, (1) implies the claimed inequality. □

3. CONCLUDING REMARKS. To obtain a lower bound as sharp as Zimmert's one can proceed as above, but with a variant Θ_1 of the function Θ. Namely, fix a real number $s > 1$, and define $\Theta_1(x)$ by the same series as $\Theta(x)$ but with the term

$$\exp\left(-\sum |\alpha|_j x_j^{2/n_j}\right)$$

replaced by

$$\prod_{j=1}^{r} f_j(|\alpha|_j^{n_j} x_j),$$

where

$$f_j(t) = \begin{cases} 2^{s-1}\Gamma(s) J_{s-1}(t)/t^{s-1} & \text{for real } |\cdot|_j \\ 2^{2s-1}\Gamma(2s) J_{2s-1}(\sqrt{t})/t^{\frac{2s-1}{2}} & \text{for complex } |\cdot|_j . \end{cases}$$

Here $J_s(t)$ denotes the J-Bessel function of order s. Then $(1 + \Theta_1)\delta$ is still increasing in each argument, as can be proved by Poisson summation. The integral of $\Theta_1 \delta^u$, taken over G/V for any $1 < u < s + \frac{1}{2}$, equals $Z_1(u)$ where $Z_1(u)$ is defined as $Z(u)$ but with $\gamma(u)$ replaced by

$$\gamma_1(u) = \left(2^{u-1}\Gamma(s)\Gamma\left(\frac{u}{2}\right)/\Gamma\left(\frac{2s-u}{2}\right)\right)^{r_1} \left(2^{2u}\Gamma(2s)\Gamma(u)/\Gamma(2s-u)\right)^{r_2}.$$

The same reasoning as above will lead then to a lower bound for R/w as in the theorem, but with γ replaced by γ_1.

The inequality (1) remains true if one replaces $Z(s)$ by the product of any partial Dedekind zeta function of K and $\gamma(s)$. It also remains true if $Z(s)$ is replaced by the (proper) Dedekind zeta function and R/w is multiplied by the class number of K. As was pointed out by W. Kohnen it might be interesting to study these inequalities for special classes of fields like, for instance, abelian fields, where the Dedekind zeta function is of a rather simple shape.

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Nils-Peter Skoruppa

Max-Planck-Institut für Mathematik
Gottfried-Claren-Strasse 26
D-W-5300 Bonn 3, FRG

E-mail address: nils@mpim-bonn.mpg.de