10. What is a unit theorem?

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for all $n \gg 0$. One then has an exact sequence

$$0 \to SL(\Lambda)/E(\Lambda) \to K_1(\Lambda) \to K_1(A)\ ,$$

and it remains to show that $K_1(\Lambda) \to K_1(A)$ has finite kernel ([Ba 1], 19.12). This implies the lemma.

We have presented Bass' theorem here because it can be viewed as an extension of Dirichlet's unit theorem. For more results on K_1 of orders, we refer the reader to [CR 2, Ch. 5]. This chapter also contains a simplified proof of Bass' theorem.

10. What is a unit theorem?

In the search for the — still missing — "basic structure theorem for units of orders" it seems natural to keep Dirichlet's theorem as our landmark; it gives in fact a presentation for all commutative unit groups. However, if we muster the small list of other cases in which explicit presentations have been obtained so far, and if we realize the comparatively elementary character of these examples, we have to admit that going for presentations is somehow utopian. Worse still, it might even be inadequate; as the general insolvability of Dehn's problems shows, we can never be sure that a presentation, obtained somehow, gives us the "right" information. For example, how could the congruence property be checked from a presentation? What then, it will now be objected, is the aim of our research? This is certainly not the place to dwell in considerations in the manner of ordinary language philosophy, but the reader may find it fruitful to ask himself what he means by saying "I know a certain group" or "I know the structure of that group". Surely we know $SL_2(\mathbf{Z})$ better than any other noncommutative unit group, but we will never know everything about it (and hence about groups containing it) because this would include knowledge of all finitely generated groups.

Leaving aside philosophy, let us try to specify what should be expected from a "general unit theorem". Unable, of course, to presume its content, we may be allowed to sketch a list of desiderata.

Let A be simple. The unit theorem should deal with torsion free subgroups of finite index of $S\Gamma$ for arbitrary Λ; such groups may be called "*generic unit groups of A*". The set of generic unit groups is closed under intersections since any two are commensurable. Naively, a unit theorem for A consists in *the definition, in purely group theoretical terms, of a class of groups $\mathscr{C}(A)$ such that almost all generic unit groups of A are members of $\mathscr{C}(A)$.*

Of course the elements of $\mathscr{C}(A)$ must have all the properties we have established for the $S\Gamma$; in particular they must be finitely presented and of cohomological dimension $r(SA) - n + 1$. They should be parametrized by the numerical invariants of A plus a parameter accounting for the index. By numerical invariants I mean the various degrees involved as well as $r(SA)$, discr Λ for Λ maximal, perhaps even class numbers and Hasse invariants. The smaller $\mathscr{C}(A)$ the better the unit theorem; optimally, $\mathscr{C}(A)$ consists — perhaps up to finitely many exceptions — of the generic unit groups of A. Our two examples are $A = M_2(\mathbf{Q})$, in which $\mathscr{C}(A)$ consists of the finitely generated free groups, and A = indefinite quaternion skewfield over $\mathbf{Q}$, in which $\mathscr{C}(A)$ consists of the fundamental groups of closed oriented surfaces.

One should realize that the existence of a definition of $\mathscr{C}(A)$ independent of A is in no way guaranteed, in other words, that there may be no pre-existing group theoretical terms by which the generic unit groups of A can be characterized. This would mean that there are algebras (presumably skewfields) which produce group — theoretical features not available from anywhere else, at least not with lesser complexity. The simplicity of the examples is surely misleading. But this may be a question for logicians and complexity theorists rather than for an "ordinary" mathematician.

Given A, we would like to distinguish in $\mathscr{C}(A)$ the maximal generic unit groups. For $A = M_2(\mathbf{Q})$, one is a free group of rank 2, occurring as the commutator group of $SL_2(\mathbf{Z})$. (I don't know whether or not all maximal torsion free subgroups of $SL_2(\mathbf{Z})$ are free of rank 2).

Given A_1 and A_2, we would like to decide whether or not they share a generic unit group (and hence infinitely many). In the number field case the unit rank is a rather weak invariant. In contrast to this, $SL_2(\mathbf{Z})$ is unique, as we have seen in section 7. In the quaternion case, there are coincidences (see the end of [E 1]).

Traditionally the geometry connected with the unit groups was considered more important than the groups themselves. Paying tribute to this view we could formulate geometric analoques to the above questions. Let SG be the elements of $A_{\mathbf{R}}^{\times}$ of reduced norm one, $C \subset SG$ a maximal compact subgroup. For generic $\Delta \subset SG$ put

$$X(\Delta) = C \backslash SG / \Delta \, .$$

Then the overall program would be *to study the manifolds* $X(\Delta)$. This is surely the most ambitious part, and pointing to the vast amount of work which has been and is currently devoted to the simplest non-settled $S\Gamma$, the Hilbert modular groups, one might criticize this laconic formulation as all too naive. (The reader who wants to get an impression of the world of mathematics meeting here should have a glance to the volume [Ge]). On the other hand, being content with subgroups of finite index, we avoid the complications arising from the torsion in $S\Gamma$. It is also conceivable that the projective system of all $X(\Delta)$ and its limit is the appropriate subject of our hypothetical "basic unit theorem". Again it can be asked what is meant by "knowing a space". A "space presentation", as analogue to a group presentation, could be an explicit cell structure; this has been obtained in a few cases. But here as elsewhere in mathematics one cannot hope to get "everything explicit"; the real problem is to define the significant invariants and to understand their mutual relations. If there is a single theorem deserving the name "General Unit Theorem" it will probably relate arithmetical and geometrical invariants.

Of particular importance will be those of cohomological origin. Note that in our two examples the decisive invariant (rank and genus, respectively) is nothing more than a first Betti number. It is clear that things wont't be so easy generally. But at least the following result deserves to be mentioned here: for generic unit groups $\Delta_1 \subset \Delta_2$ one has

$$|\Delta_2 : \Delta_1| \chi(\Delta_2) = \chi(\Delta_1) ,$$

χ denoting Euler characteristics. (See [Se 3], p. 86). If these don't vanish, this is an index formula, generalizing the Nielsen-Schreier formula

$$|\Delta_2 : \Delta_1| (rk\Delta_2 - 1) = (rk\Delta_1 - 1)$$

for $A = M_2(\mathbf{Q})$ and the Riemann-Hurwitz formula

$$|\Delta_2 : \Delta_1| (g(\Delta_2) - 1) = (g(\Delta_1) - 1)$$

in the quaternion case (g denoting genus).

Finally, let us muster the algebras with small $r(SA)$ and see what could be done next. We exclude $A = K$; that is $k = ns > 1$ formula (5). Note that $r_1'' > 0$ implies s even, in particular > 0. $r(SA) = 0$ occurs only for $r_1' = r_2 = 0$, $n = 1$, $s = 2$, r_1'' arbitrary. This means that A is a totally definite quaternion skew field, and we have noted already that $S\Gamma$ is finite in such cases which we therefore consider as settled. (It is interesting to note that these algebras are exceptional in other respects, too — to "compensate" for the easy unit theory, their module theory is more difficult.) $r(SA) = 1$ is not possible (as the reader should check from (5). (Conceptual explanation: if $r(SA) = 1$, then a generic unit group would be the fundamental group of a one dimensional manifold, hence abelian. On the other hand, if it is infinite, it is Zariski dense in SG, by a theorem of Borel ([P]), Th. 1.5). Thus, A would be commutative). If $r(SA) = 2$, by necessity $r_2 = 0$, $ns = 2$, $r_1' > 0$. We may have $n = 2$, $s = 1$ and consequently $r_1'' = 0$; this gives $A = M_2(\mathbf{Q})$; or $n = 1 = r_1'$, $s = 2$ and r_1'' arbitrary. Then A is a quaternion skew field over a totally real K ramified at all but one of the infinite primes of K. (Eichler's case is $r_1'' = 0$.) The image of the $S\Gamma$ in $PSL_2(\mathbf{R})$ are special Fuchsian groups characterized among all Fuchsian groups by the behavior of their traces [Ta]. Now finitely generated Fuchsian groups have a standard presentation given by their "signature" (see [F], p. 37). It should be possible to calculate the signatures in terms of the arithmetic invariants of A, generalizing Eichler's result. $r(SA) = 3$ requires $r_1' = 0$, $r_2 = 1$, $ns = 2$, r_1'' arbitrary. $s = 1$, $r_1'' = 0$ is the case of the Bianchi groups. For $n = 1$, $s = 2$, r_1'' arbitrary A is a quaternion skewfield over a field K with one complex embedding, ramified at all real infinite primes of K. The images of $S\Gamma$ in $PSL_2[\mathbf{C}]$ are special Kleinian groups, acting discontinuously on hyperbolic 3-space. It should be possible to treat them as the Bianchi groups. Similarly with $r(SA) = 4$ we encounter the Hilbert modular groups, but also quaternion skewfields over totally real fields ramified at all but two of the infinite primes ($r_2 = 0$, $r_1' = 1$, $s = 2$, $n = 1$, r_1'' arbitrary). At least if $r_1'' = 0$ (so A is ramified only at finite primes) the skewfield case can hardly be of more complicated structure than the matrix case; it should be even easier in view of the fact that bounded fundamental domains exist. That they have been studied much less must probably be ascribed to the circumstance that it is not so easy to write down units in skewfields. This brings us to our last point namely the

PROBLEM. *Give an algorithm which constructs generators of a subgroup of finite index of* $S\Gamma$.

This problem has in principle been solved by Grunewald and Segal ([GS], Algorithm B). As so many other results of this survey, their algorithm applies to arithmetic groups and is, as the authors point out, even in this generality not best possible. Bringing in, in the case of units of orders, the underlying ring structure, one should be able to give manageable procedures. The main interest lies in the case $A = D$ which seems to be untouched (in this respect). Since every $x \in \Gamma^\times \backslash R^\times$ generates an extension of number fields $K(x) \mid K$, the methods of computational number theory will enter the game. In view of this, it will be of advantage that we may choose Λ to be a cyclic crossed product order.