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Autor(en): **Jones, V.F.R.**

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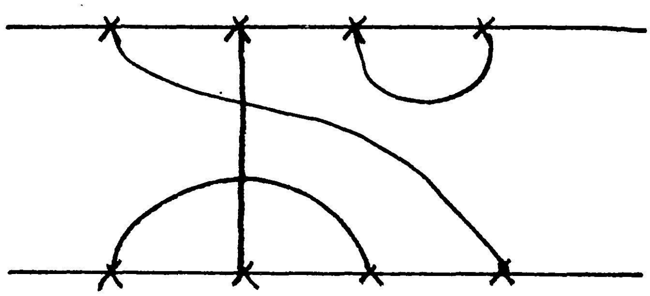
A QUOTIENT OF THE AFFINE HECKE ALGEBRA IN THE BRAUER ALGEBRA

by V.F.R. JONES[1])

ABSTRACT. The structure of a certain subalgebra of Brauer's centralizer algebra is given for all values of the parameter for which it is semisimple. The algebra admits a trace functional whose weights on the simple components of the algebra are calculated. The algebra may be exhibited as a quotient of the affine Hecke algebra of type $\tilde{A}_n$, using generators and relations.

0. INTRODUCTION

Brauer's centralizer algebra is defined abstractly as having a basis of diagrams as below, multiplied in a rather obvious fashion (see [B]) which involves a parameter δ. This algebra is an abstract model for the commutants of the tensor powers of the defining representations of (odd) orthogonal and symplectic groups, the parameter δ in the algebra being $\pm$ the dimension of the space. For generic values of the parameter the Brauer algebra is semisimple and its structure is known (see [W], [HW]).



A basis element of the Brauer algebra on four points.

The Brauer algebra on n points contains certain subalgebras defined by "topological" conditions. The most obvious is the so-called Temperley-Lieb

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algebra spanned by diagrams that are planar (have no crossings). It is a much smaller algebra than the Brauer algebra and its structure is easy and extremely well known, at least when semisimple (see [GHJ], [GW]). In this paper we analyse a slightly larger subalgebra of the Brauer algebra, namely that spanned by diagrams that can be realised without crossings in an annulus; see below.

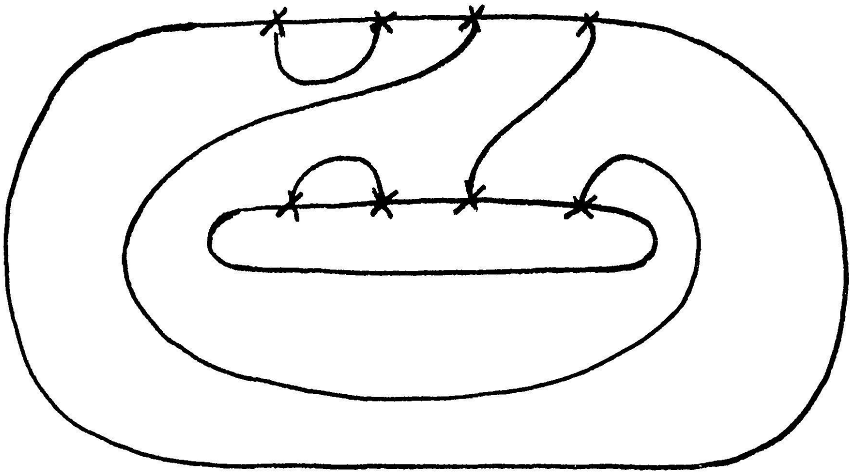


FIGURE 0: An annular diagram.

If we call β the Brauer representation into the commutant of $O(k)$ on $\otimes^n \mathbf{C}^k$, it is known that the restriction of β to the Temperley-Lieb algebra is faithful for $k \geqslant 2$. We show that the restriction to our annular algebra is faithful for $k \geqslant 3$ (but not for $k = 2$). This shows that the annular algebra is generically semisimple. For a value of δ for which the annular algebra is semisimple we show that the irreducible representations of the annular algebra (over $\mathbf{C}$) are parametrised by

(1) an integer t, $0 \leqslant t \leqslant n$ with $n + t$ even,

(2) a t-th root of unity.

The dimension of the corresponding representation, called $\pi_{t,\omega}$, is $\binom{n}{\frac{n-t}{2}}$ for $t > 0$ and $\frac{1}{\frac{n}{2}+1}\binom{n}{\frac{n}{2}}$ for $t = 0$ so that the dimension of the algebra is

$$\left\{\frac{1}{\frac{n}{2}+1}\binom{n}{\frac{n}{2}}\right\}^2 + \sum_{\substack{t+n \text{ even} \\ 0 < t \leqslant n}} t\binom{n}{\frac{n-t}{2}}^2,$$

which is the same as the number of annular diagrams.

There is a trace functional on the annular algebra which can be defined either via the Brauer representation or by counting the number of closed loops when a diagram is closed by identifying the inside and outside circles. This trace is determined by its values on idempotents $p_{t,\omega}$ whose principal left ideals define the representations $\pi_{t,\omega}$. Call the trace of such an idempotent $\mathcal{M}(\omega, t)$. Then we show that $t\mathcal{M}(\omega, t)$ is the integer valued polynomial in δ given by $2\sum_{r=0}^{t-1}\omega^r\cos(GCD(t,r)\theta)$, $2\cos\theta = \delta$. These also give the multiplicities of $\pi_{t,\omega}$ in β. In appendix 1 we give a table of the dimensions of the irreducible representations of the annular algebra for $n > 9$, and a table of the weights of the above trace.

The key to the analysis of the annular algebra is the observation that it is filtered by ideals corresponding to the number of "through-strings". This idea occurs in [B] (see [HW]) and we have taken the terminology from [MW]. For us it was inspired by the first way of counting annular diagrams presented in § 1, which was discovered by F. Jaeger, to whom the author is most grateful. It was necessary to use this "through string" technique, which is, technologically speaking, a backwards step from Wenzl's paper [W], since the annular algebras are *not* unitally included in one another. This means that the "basic construction" technique is not available.

A special system of generators of the annular algebra exhibits it as a quotient of the affine Hecke algebra of type $\tilde{A}_n$ with parameter $q(\delta = 2 + q + q^{-1})$ (see remarks after Theorem 2.8).

The original motivation for this work was to help calculate centraliser towers in subfactors. Given an extremal (see [PP]) subfactor N of a II_1 factor M, one forms the tower M_i as in [J] with $M_{i+1} = \langle M_i, e_{i+1}\rangle$, $M_0 = M$, $M_{-1} = N$. Then there is an action of an affine Hecke quotient on $N' \cap M_n$ according to the generators $f_1, f_2, \ldots, f_{2n+2}$ defined by:

$$f_i = \begin{cases} \text{left multiplication by } e_i, & 1 \leqslant i \leqslant n \\ E_{M_{n-1}} \text{ (conditional expectation)}, & \text{for } i = n+1 \\ \text{right multiplication by } e_{2n-i+2} & n+2 \leqslant i \leqslant 2n+1 \\ E_{M'}, & \text{for } i = 2n+2. \end{cases}$$

They satisfy: $f_i^2 = f_i$, $f_i f_{i\pm 1} f_i = \tau f_i$, $f_i f_j = f_j f_i$ if $j \neq i \pm 1$, where the indices are taken in $\mathbf{Z}/(2n+2)\mathbf{Z}$, and where τ^{-1} is the index of N in M. The result of this paper gives the structure of the algebra in the example $M = N \otimes M_k(\mathbf{C})$ — it is the oriented subalgebra of the annular algebra. In general the affine Hecke modules occurring in $N' \cap M_n$ depend sensitively on the subfactor. We will present more results on this elsewhere.

I would also like to thank D. Levy (see [Le]) and C. Cibils for conversations and S. Eliahou for some useful computer calculations.

1. Counting diagrams

Definition 1.1. An (a, b) *diagram D* will be a partition of the union of a set of size "a" and a set of size "b" into subsets of size 2 (so $a + b$ is even). If the set of size "a" consists of points on one line and the set of size "b" consists of points on another, the diagram may be represented pictorially as below:

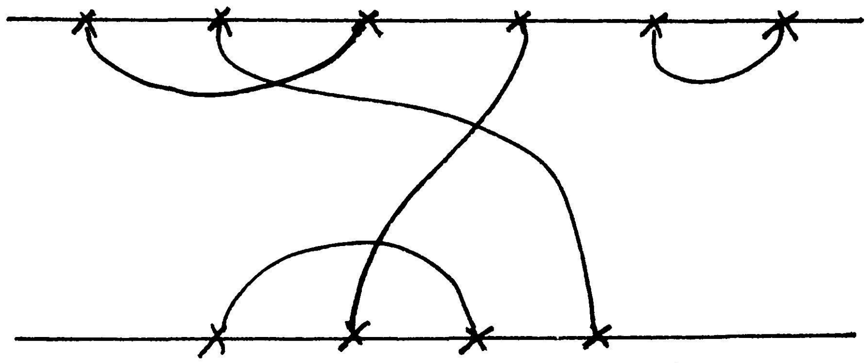


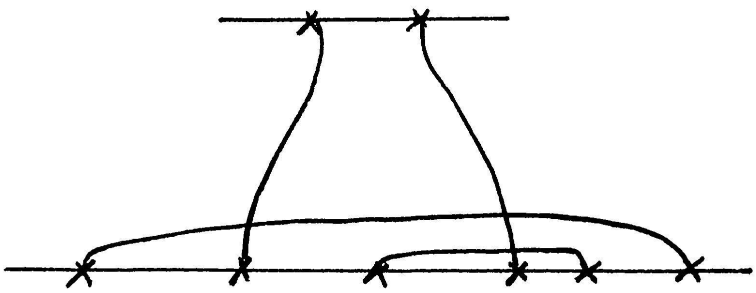
FIGURE 1.2: A (4, 6) diagram α.

The set of all (a, b) diagrams will be denoted $\mathscr{D}(a, b)$.

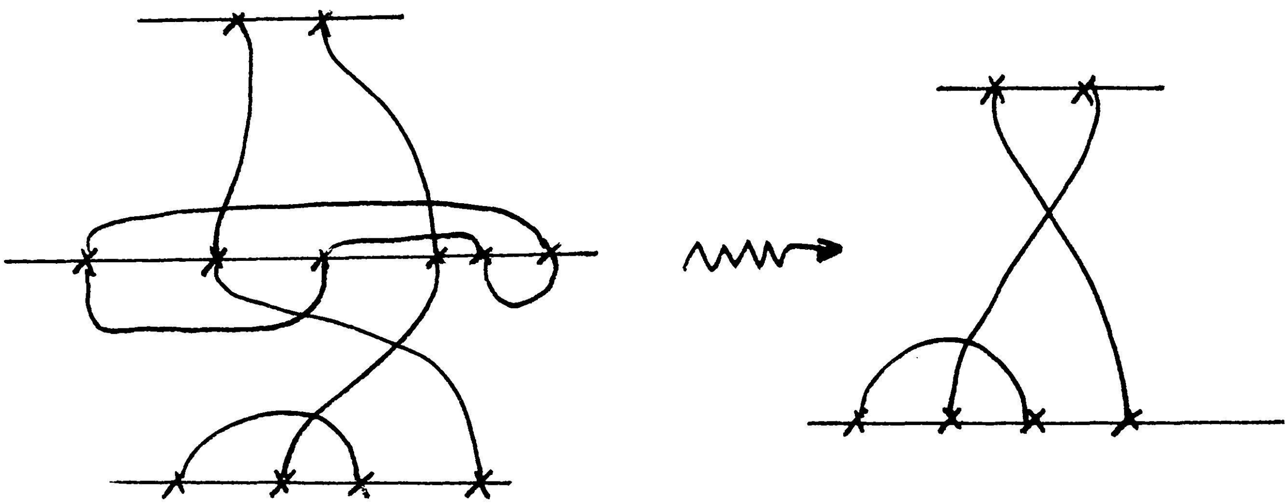
Definition 1.3. If $\alpha \in \mathscr{D}(a, b)$ a *through-string* for the pictorial representation of α is a line going from one of the top points to one of the bottom ones. The number of through strings of α will be written $t(\alpha)$.

If there is some way of identifying sets of a given size (such as if they are points on a line or a circle), we may define an associative rule $(\alpha, \beta) \mapsto \alpha \circ \beta$ allowing one to multiply a $\mathscr{D}(a, b)$ diagram by a $\mathscr{D}(b, c)$ diagram to get a $\mathscr{D}(a, c)$ diagram. No doubt the neatest way to define all this is in terms of categories but we prefer to remain extremely concrete so we define the associative rule pictorially by concatenation of diagrams, as below.

FIGURE 1.4. With α as in Figure 1.1 and $\beta \in \mathscr{D}(6, 2)$ as follows,



$\alpha \circ \beta$ is formed as follows:



When two diagrams α and β are multiplied, a certain number of closed loops are formed which are forgotten in the product $\alpha \circ \beta$. We call $n(\alpha, \beta)$ the number of closed loops.

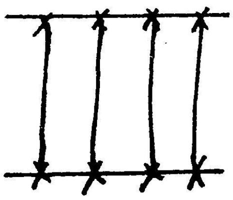
LEMMA 1.5. *If* $\alpha \in \mathscr{D}(a, b)$, $\beta \in \mathscr{D}(b, c)$, $\gamma \in \mathscr{D}(c, d)$,

(i) $n(\alpha, \beta) + n(\alpha \circ \beta, \gamma) = n(\beta, \gamma) + n(\alpha, \beta \circ \gamma)$

(ii) $t(\alpha \circ \beta) \leqslant \min\{t(\alpha), t(\beta)\}$

Proof. (i) Both sides count the number of closed loops in the figure obtained by concatenating the figures of α, β and γ.

(ii) Obvious.

The diagram  in $\mathscr{D}(n, n)$ is obviously an identity for $\circ$ and will be denoted 1.

Definition 1.6. We will say that a diagram $D \in \mathscr{D}(a, b)$ is *planar* if a figure representing the diagram has no crossings and all the connecting lines do not leave the strip in the plane defined by the top and bottom lines.

We will say that D is *annular* if the $a + b$ points are on the inside and outside of an annulus with all connecting lines inside the annulus, and without crossings as in Figure 1.7.

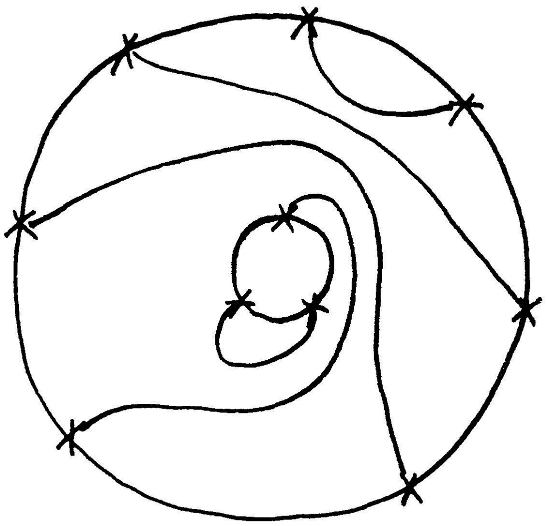


FIGURE 1.7: An annular (3, 7) diagram with 1 through-string.

The set of planar (a, b) diagrams will be denoted $\mathscr{P}(a, b)$ and those having t through-strings $\mathscr{P}(a, b; t)$, and the set of annular (a, b) diagrams $\mathscr{A}(a, b)$ and those having t through-strings $\mathscr{A}(a, b; t)$. It is well known that the set $\mathscr{P}(n, n)$ has order $\operatorname{cat}(n) = \frac{1}{n+1}\binom{2n}{n}$. We shall count $\mathscr{A}(a, b)$.

It will be convenient to fix two points on the inside and outside circle respectively, call them $*$, and suppose they lie on a vertical line thus:

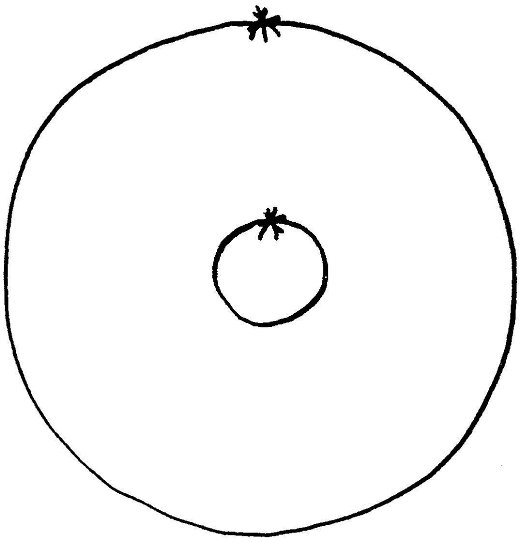


FIGURE 1.8

Because of applications to subfactors we shall be interested in the case $\mathscr{A}(n, n)$, for which we say that a diagram is *oriented* if the curves joining points may be oriented so that, at the boundary points, they point alternately inward and outward, with the top $*$ pointing outward and the bottom $*$ pointing inward. Thus Figure 1.9 is a figure of an oriented diagram.

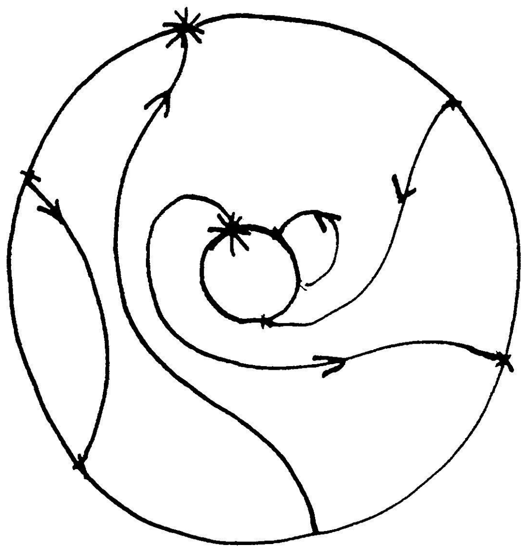


FIGURE 1.9

The set of oriented diagrams will be denoted $\vec{\mathscr{A}}(n, n)$. Note that any planar diagram is oriented. There are 22 oriented diagrams in $\mathscr{A}(4, 4)$ and 40 elements altogether, which we enumerate in appendix 3 as they should be quite useful in understanding the rest of the paper.

The difference between the planar and annular cases is that the curves in the diagram may go round the circle. It is easy to guess that the element we are about to define in $\mathscr{A}(n, n; n)$ will have a significant role to play.

Definition 1.10. The element $u \in \mathscr{A}(n, n; n)$ is the diagram of a cyclic permutation represented by the figure below.

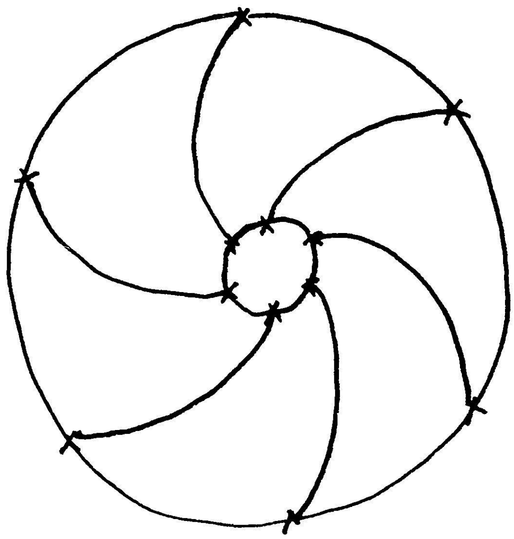


FIGURE 1.11: $u \in \mathscr{A}(6, 6; 6)$

PROPOSITION 1.12. *For the semigroup structure on* $\mathscr{A}(n, n)$ *determined by* o, $u^n = 1$.

Proof. Although the figure represented by u^n has a 360° twist, the corresponding diagram is the identity. □

We now want to count annular diagrams. We begin with a preliminary result which is well known but we include a proof for the convenience of the reader.

LEMMA 1.13. *Let* $\left\{ {n \atop p} \right\} = |\mathscr{P}(n,p;p)|$, *the number of planar* (n,p) *diagrams with* p *through-strings. Then*

$$\left\{ {n \atop p} \right\} = \binom{n}{\frac{n-p}{2}} - \binom{n}{\frac{n-p-2}{2}} = \frac{p+1}{n+1}\binom{n+1}{\frac{n-p}{2}}.$$

Proof. Consider an element of $\mathscr{P}(n+1, p-1; p-1)$. The leftmost of the bottom $n+1$ points, call it x, is either connected to the top or the bottom. If it is connected to the top it must be to the leftmost of those $p-1$ points since there are $p-1$ through-strings. There are thus $\left\{ {n \atop p-2} \right\}$ such. If x is connected to the bottom, one may move x to the top to obtain an element of $\mathscr{P}(n,p)$ as below:

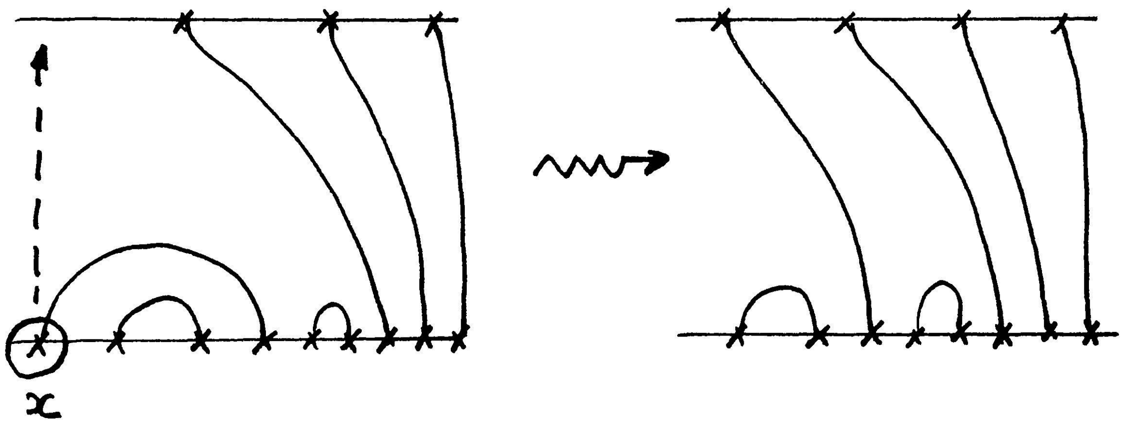


FIGURE 1.14

The process may clearly be reversed so that there are $\left\{ {n \atop p} \right\}$ such and

$$\left\{ {n+1 \atop p-1} \right\} = \left\{ {n \atop p} \right\} + \left\{ {n \atop p-2} \right\}.$$

Putting $\left\langle {n \atop p} \right\rangle = \left\{ {n \atop n-2p} \right\}$ for $p = 0, 1, \ldots \left[\frac{n}{2}\right]$ we see that

$$\left\langle {n+1 \atop p} \right\rangle = \left\langle {n \atop p} \right\rangle + \left\langle {n \atop p-1} \right\rangle .$$

But the same recursion relations and boundary conditions are satisfied by $\binom{n}{p} - \binom{n}{p-1}$. So

$$\left\langle {n \atop p} \right\rangle = \binom{n}{p} - \binom{n}{p-1} \quad \text{and} \quad \left\{ {n \atop p} \right\} = \left\langle {n \atop \frac{n-p}{2}} \right\rangle = \binom{n}{\frac{n-p}{2}} - \binom{n}{\frac{n-p-2}{2}} .$$

Also note $\binom{n}{k} - \binom{n}{k-1} = \dfrac{n-2k+1}{n+1}\binom{n+1}{k}$. □

Corollary 1.15. $|\mathscr{A}(n,p;p)| = p\binom{n}{\frac{n-p}{2}}$ *for* $p \neq 0$.

Proof. Fix one of the p outside points. There are n ways to connect it to the inside. Once connected, one may cut the annulus open along that string and one is in the planar situation so

$$|\mathscr{A}(n,p;p)| = n\left\{ {n-1 \atop p-1} \right\} = \frac{np}{n}\binom{n}{\frac{n-p}{2}} = p\binom{n}{\frac{n-p}{2}} . \quad \square$$

Corollary 1.16. $|\mathscr{A}(p,q;t)| = t\binom{p}{\frac{p-t}{2}}\binom{q}{\frac{q-t}{2}}$ *for* $t > 0$.

Proof. Given an annular diagram D with $t(D) = t$, one may push all the curves connecting inner (resp. outer) boundary points to lie within a small neighborhood of the inner (resp. outer) boundary circles. Then D may be cut with a third circle, concentric to the others, which meets only the through-strings. Thus we can write $D = D_1 \circ D_2$ with $D_1 \in \mathscr{A}(p,t;t)$, $D_2 \in \mathscr{A}(t,q;t)$. Moreover given the pair D_1, D_2, it is immediate that any other pair E_1, E_2 with $E_1 \circ E_2 = D$ is of the form $E_i = D_1 \circ u^k$, $E_2 = u^{-k} \circ D_2$ for some $k = 1, 2, \ldots, t$ ($u \in \mathscr{A}(t,t)$ as in Definition 1.10). And all t such pairs (E_1, E_2) are clearly distinct. Thus

$$|\mathscr{A}(p,q;t)| = \frac{1}{t}|\mathscr{A}(p,t;t)||\mathscr{A}(t,q;t)| = t\binom{p}{\frac{p-t}{2}}\binom{q}{\frac{q-t}{2}} . \quad \square$$

COROLLARY 1.17.

$$|\mathscr{A}(2p,2q)| = \mathrm{cat}(p)\,\mathrm{cat}(q) + \sum_{i=1}^{\min(p,q)} 2i \binom{2q}{q-i}\binom{2p}{p-i} \quad \textit{and}$$

$$|\mathscr{A}(2p+1,2q+1)| = \sum_{i=0}^{\min(p,q)} (2i+1)\binom{2p+1}{p-i}\binom{2q+1}{q-i}.$$

Proof. The only case not covered explicitly by Corollary 1.16 is the case of zero through-strings. This can only happen if p and q are even, and then the argument of 1.16 shows that the number is $|\mathscr{A}(2p,0)\|\mathscr{A}(2q,0)| = |\mathscr{P}(2p,0)\|\mathscr{P}(2q,0)|$. □

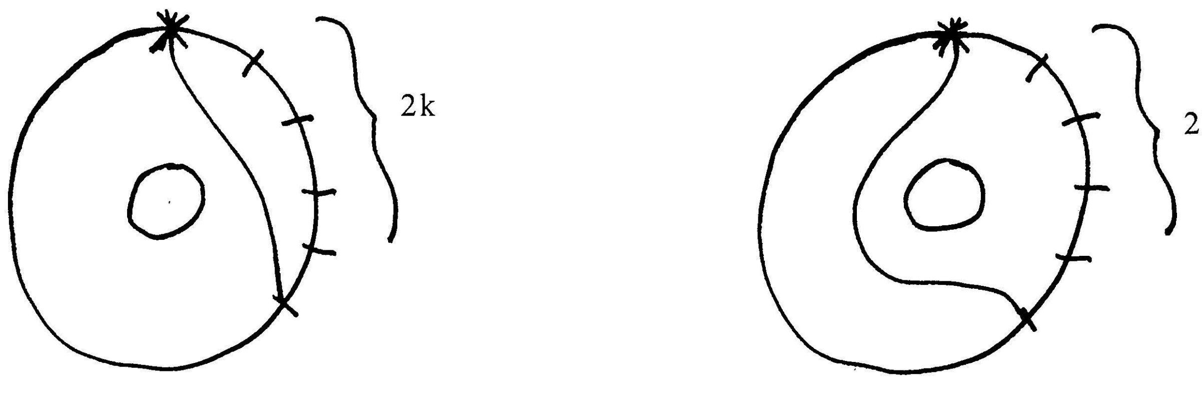
S. Eliahou has calculated $|\mathscr{A}(2p,2q)| = \frac{pq}{p+q}\binom{2p}{p}\binom{2q}{q} + \mathrm{cat}(p)\,\mathrm{cat}(q)$.

It is possible to count the elements in $\mathscr{A}(2p,2q)$ in a completely different way which we now explain. This other way will not be used, but the truth of the resulting binomial identity confirms the calculation.

To save on notation let $d_{p,q} = |\cup_{t>0}\mathscr{A}(2q,2p;t)|$ and let c_k also stand for $\mathrm{cat}(k)$.

First count all the elements of $\mathscr{A}(2q,2p)$ with (the outer) $*$ connected to the inner circle. Clearly $*$ can be connected to $2q$ points and, once connected, there are $\mathrm{cat}(p+q-1)$ ways of completing the diagram. There are thus $2q\,\mathrm{cat}(p+q-1)$ such.

Now assume $*$ is connected to the outside. Then D is of the form (a) or (b) in Figure 1.18:



(a) (b)

FIGURE 1.18

Thus for the $p-1$ possible points to which $*$ is connected we must count the two possibilities (which are distinct since there is at least one through-string), which are clearly $d_{p-k-1,q}c_k$ and $c_{p-k-1}d_{k,q}$. Altogether we get $d_{p,q} = q\,\mathrm{cat}(p+q-1) + c_0 d_{p-1,q} + c_2 d_{p-2,q} + \cdots + c_{p-2} d_{1,q} + d_{1,q} c_{p-2} + \cdots + c_0 d_{p-1,q}$ or

$$d_{p+1,q} = q\,\mathrm{cat}(p+q) + 2\sum_{i=0}^{p-1} c_i d_{p-i,q}$$

To get an explicit formula for $d_{p,q}$, we use generating functions. Let $\mathrm{CAT}(x) = \sum_{n=0}^{\infty} \mathrm{cat}(n) x^n = \frac{1-\sqrt{1-4x}}{2x}$ and $f_q(x) = \sum_{r=0}^{\infty} d_{r+1,q} x^r$. Then

$$f_q(x) = q\sum_{n=0}^{\infty} \mathrm{cat}(q+n)x^n + 2x\mathrm{CAT}(x) f_q(x)$$

so that

$$\begin{aligned} f_q(x) &= q(1-4x)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \mathrm{cat}(q+n)x^n \\ &= q\left(\sum_{r=0}^{\infty}\binom{2r}{r}x^r\right)\left(\sum_{n=0}^{\infty}\frac{1}{n+q+1}\binom{2n+2q}{n+q}x^n\right). \end{aligned}$$

Equating coefficients we see we have proven:

LEMMA 1.19.

$$\begin{aligned} |\mathscr{A}(2p, 2q)| - \mathrm{cat}(p)\mathrm{cat}(q) &= 2q\sum_{n=0}^{p-1}\frac{1}{p+q-n}\binom{2n}{n}\binom{2p+2+2q-2n-2}{p+q-n-1} \\ &= \sum_{t=1}^{\min(p,q)} 2t\binom{2p}{p-t}\binom{2q}{q-t} \end{aligned}$$

Finally we observe that oriented diagrams are easily counted from nonoriented ones.

LEMMA 1.20. *If* p *is even and* $u \in \mathscr{A}(p,p)$ *is as in 1.10, then for* $t > 0$, $\alpha \mapsto u\alpha$ *is a bijection between* $\overrightarrow{\mathscr{A}(p,q;t)}$ *and unoriented elements of* $\mathscr{A}(p,q;t)$. *Thus* $|\mathscr{A}(p,q;t)| = 2|\mathscr{A}(p,q;t)|$.

Proof. It suffices to show that, if α is oriented, $u\alpha$ is not, and vice versa. The first assertion is obvious. So if α is not oriented, choose a non-oriented through-string. The same string extended through $u\alpha$ is then oriented. Cutting along that string we are in the planar situation which is necessarily oriented for obvious parity reasons. $\square$

2. The abstract algebras

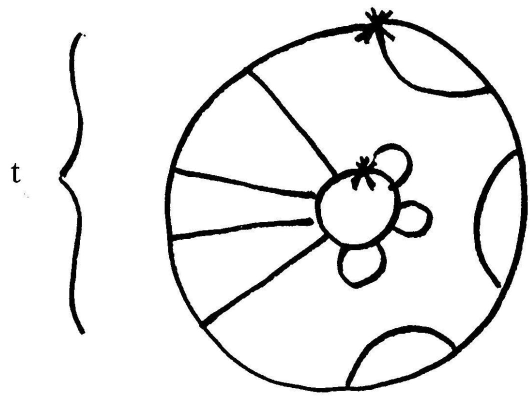
The (abstract) Brauer algebra with parameter $\delta \in \mathbf{C}$, $B(n, \delta)$, is the algebra with basis the set of all (n, n)-diagrams and multiplication law $\alpha\beta = \delta^{n(\alpha,\beta)}\,\alpha \circ \beta$. We could say it is the twisted monoid group algebra for the monoid $(D(n, n), \circ, 1)$ and the cocycle δ^n. We have thus at our disposition two other series of abstract algebras with parameter, subalgebras of the Brauer algebra:

$P(n, \delta) =$ The subalgebra spanned by planar diagrams also called the Temperley-Lieb algebra $TL(n, \delta)$, in fact invented as diagrams by Kauffmann ([K]).

$A(n, \delta) =$ The subalgebra spanned by annular diagrams.

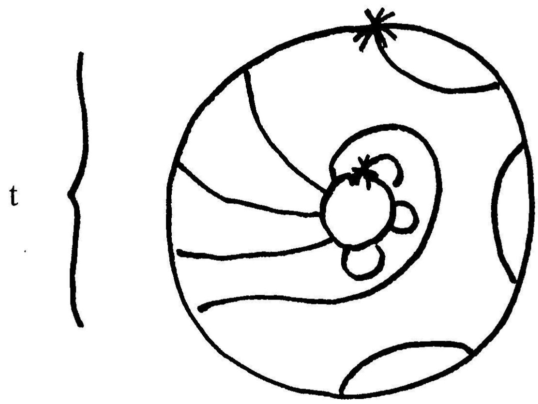
The structure of the Brauer algebra has been studied extensively. See [W], [HW] for much information, and $P(n, \delta)$ is particularly well understood (see [GW], [GHJ]). In this section we will give the structure of $A(n, \delta)$ whenever it is semisimple (over $\mathbf{C}$). It will be worthwhile to call the algebra simply $A(\mathrm{n})$ in this section since we will only consider a fixed δ $(\neq 0)$.

Definition 2.1. (i) We call $E(n, t)$ the diagram (in $\mathscr{A}(n, n; t)$)



(so that $E(n, n) = 1$).

(ii) We call $V(n, t)$ the diagram (in $\mathscr{A}(n, n; t)$)



(so that $u = V(n, n)$ and $E(n, 0) = V(n, 0)$).

Note: the role of $*$ is unimportant, it serves only to have a well defined element.

LEMMA 2.2. *Let* $e_t \in A(n)$ *be* $\delta^{-\left(\frac{n-1}{2}\right)} E(n,t)$ *and* $v_t \in A(n)$ *be* $\delta^{-\left(\frac{n-t}{2}\right)} V(n,t)$. *Then*

(i) $e_t^2 = e_t$.

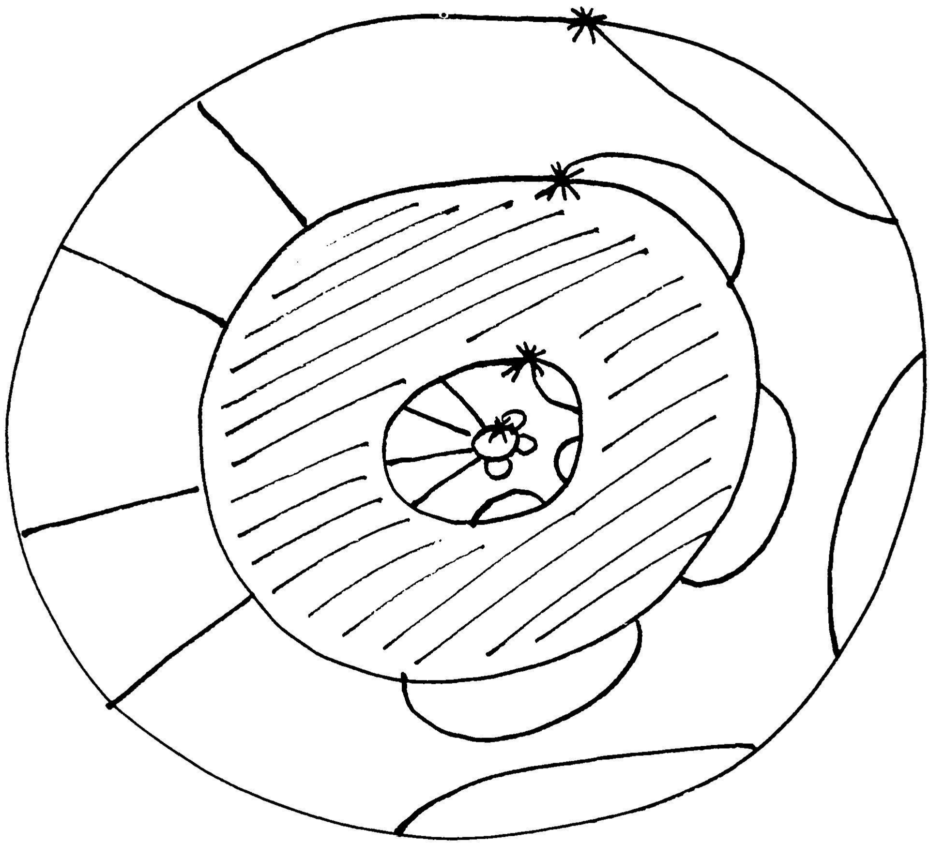
(ii) $(v_t)^t = e_t$ *(so* $e_t v_t = v_t e_t$*).*

(iii) $E(n,t) \circ \mathscr{A}(n,n) \circ E(n,t) \subset \cup_{j<t} \mathscr{A}(n,n;j) \cup \{V(n,n)^k \mid k = 0, 1, 2, \dots, t-1\}$.

(iv) *If* $D \in \mathscr{A}(n,n;t)$, *there are* D_1 *and* D_2 *in* $\mathscr{A}(n,n,t)$ *with* $D = D_1 \circ E(n,t) \circ D_2$.

Proof. (i) and (ii) are evident from diagrams and the multiplication structure in $A(n)$.

(iii) For any D in $\mathscr{A}(n,n)$, $x = E(n,t) \circ D \circ E(n,t)$ is as below.



where there is any annular diagram in the intermediate annulus (shaded). But we see that if x has t through-strings, the intermediate system must connect all of the outer through-strings to one of the inner ones. Once one connection is fixed, all the others must follow in cyclic order, so x is a power of V (with respect to $\circ$).

(iv) As in the proof of Corollary 1.16, we may write $D = E_1 \circ E_2$ with $E_1 \in \mathscr{A}(n,t;t)$, $E_2 \in \mathscr{A}(t,n;t)$. But then pulling the strings around in the middle and introducing $\frac{n-t}{2}$ isolated circles we see that D admits the desired decomposition. □

We proceed to determine the structure of $A(n, \delta)$ when it is semisimple. Note first that the through-strings give a filtration of $A(n)$ by ideals.

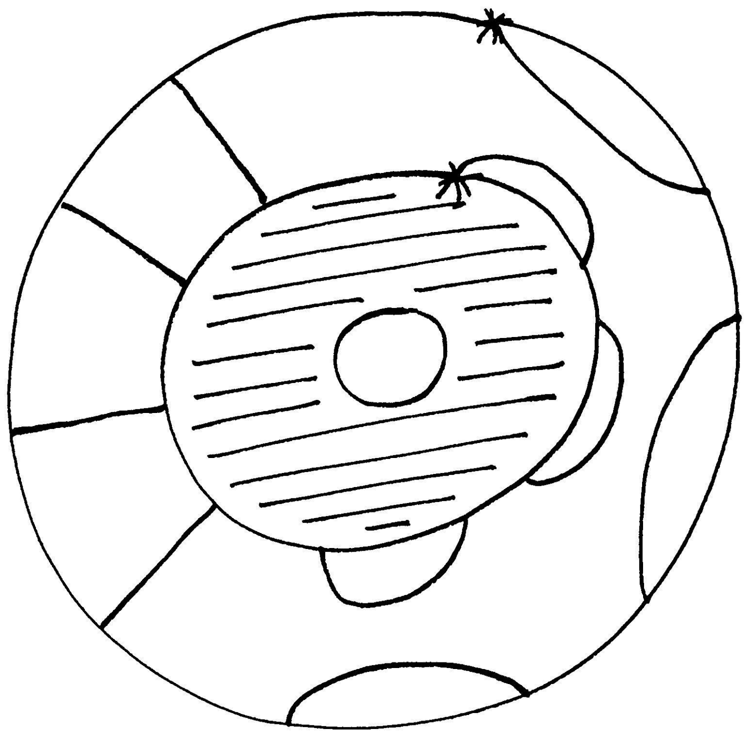
Definition 2.3. $A(n; t)$ is the two-sided ideal linearly spanned by diagrams with $\leqslant t$ through-strings.

Thus if $A(n)$ is semisimple, it is isomorphic to the direct sum $\bigoplus_{t=0}^{n} \frac{A(n;t)}{A(n;t-2)}$, and to determine its structure it suffices to determine that of the quotients, which of course are all semisimple.

THEOREM 2.4. *If* δ *is such that* $A(n, \delta)$ *is semisimple,*

$$\frac{A(n,t)}{A(n,t-2)} \cong \begin{cases} \text{A matrix algebra of size } \operatorname{cat}\left(\frac{n}{2}\right) & \text{if } t = 0 \text{ and } n \text{ even.} \\ \text{The sum of } t \text{ matrix algebras of size } \binom{n}{\frac{n-t}{2}} & \text{if } t > 0 \\ \text{(and } n - t \text{ even).} \end{cases}$$

Proof. Suppose first $t > 0$. Let A stand for $A(n, t)/A(n, t-2)$ for short and let it be isomorphic to $\bigoplus_{i=1}^{r} M_{d_i}(\mathbf{C})$. Identify elements of $A(n, t)$ with their classes modulo $A(n, t-2)$. Then by (iv) of Lemma 2.2, the 2-sided ideal generated by e_t is all of $\bigoplus_{i=1}^{r} M_{d_i}(\mathbf{C})$ so we can write $e_t = \bigoplus_{i=1}^{r} p_i$ with p_i a non-zero idempotent in each $M_{d_i}(\mathbf{C})$. But A is linearly spanned by the diagrams in $\mathscr{A}(n, n; t)$ so by (ii) and (iii) of 2.2, $e_t A e_t$ is abelian of dimension t. *Thus each of the* p_i's is a minimal idempotent, $r = t$ and of course $\sum_{i=1}^{t} d_i^2 = t \binom{n}{\frac{n-t}{2}}^2$ by (1.16). But also $\mathscr{A}(n, n; t) \circ E(n, t)$ is exactly all diagrams of the form



so that the ones representing non-zero elements of A are in bijection with $\mathscr{A}(n, t; t)$. Hence $\dim(Ae_t) = |\mathscr{A}(n, t; t)| = t\binom{n}{\frac{n-t}{2}}$. However, $(\oplus_{i=1}^{t} M_{d_i}(\mathbf{C}))\ (\oplus_{i=1}^{t} p_i)$ is a vector space of dimension $\sum_{i=1}^{t} d_i$, so we have

$$\sum_{i=1}^{t} d_i^2 = t\binom{n}{\frac{n-t}{2}} \quad \text{and} \quad \sum_{i=1}^{t} d_i = t\binom{n}{\frac{n-t}{2}}.$$

Thus each of the d_i's is equal to $\binom{n}{\frac{n-t}{2}}$ (e.g. by the "equality" case of the Cauchy Schwartz inequality $(\Sigma d_i \cdot 1) \leqslant \sqrt{\Sigma d_i^2}\sqrt{t})$. This proves the theorem for $t > 0$. The case $t = 0$ follows from the same argument, using $\dim(\mathscr{A}(n, n; 0)) = \mathrm{cat}(n)^2$ and $\dim(\mathscr{A}(n, n; 0)e_0) = \mathrm{cat}(n)$. □

Note that one could avoid the slightly clumsy Cauchy-Schwartz argument by showing that the commutant of $\mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$ is $A(n)$, which is not hard.

Remark 2.5. In fact it is clear from the proof that the algebra $e_t(A(n, t)/A(n, t-1))e_t$ is naturally isomorphic to the group algebra $\mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$, so that the various matrix algebras in $A(n, t)/A(n, t-2)$ are naturally indexed by the t-th roots of unity.

Remark 2.6. In view of 2.5, another way of stating Theorem 2.4 is to say that, if $A(n, \delta)$ is semisimple, its irreducible representations are parametrised by

(i) the number of through-strings t

(ii) a t-th root of unity ω.

Moreover the irreducible representation $\pi = \pi_{t,\omega}$ corresponding to (t, ω) is characterised by the fact that $\pi(v_t) = \omega\pi(e_t)$, and may be given quite explicitly as follows:

If W is the vector space spanned by $\mathscr{A}(t, n; t)$, W becomes an $A(n) - \mathbf{C}[\mathbf{Z}/t\mathbf{Z}]$ bimodule under the left and right action:

$$D \cdot E \cdot F = \begin{cases} D \circ E \circ F & \text{for } D \in \mathscr{A}(n, n) \text{ and } F \in \mathscr{A}(t, t; t), \text{ identified} \\ & \quad \text{with } \mathbf{Z}/t\mathbf{Z}. \\ 0 & \text{if } D \circ E \text{ has } < t \text{ through-strings.} \end{cases}$$

Then if $P_\omega = \frac{1}{t}\sum_{i=1}^{t} \omega^{-i}u^i$ (u as in 1.10), $\pi_{t,\omega}$ is left multiplication on VP_ω.

We give the structure of the subalgebra $\overrightarrow{A(n)}$ of $A(n)$ spanned by oriented diagrams. With obvious notation the result is

THEOREM 2.7. *If* δ *is such that* $\overrightarrow{A(n,\delta)}$ *is semisimple, (n even),*

$$\overrightarrow{A(n,t)}/\overrightarrow{A(n,t-2)} \cong \begin{cases} \text{A matrix algebra of size } \operatorname{cat}\binom{n}{2} & \text{if } t=0. \\ \text{The sum of } \frac{t}{2} \text{ copies of a matrix algebra} & \\ \text{of size } \binom{n}{\frac{n-t}{2}} \text{ if } t>0. & \end{cases}$$

Proof. One can simply repeat the proof of Theorem 2.4, the only difference being that the role of the element υ would be played by υ^2. One could also deduce 2.7 from 2.4 in several ways. One is to note that $\overrightarrow{\mathscr{A}(n)}$ is the fixed point algebra for an involutive automorphism of $\mathscr{A}(n)$ sending u to $-u$. Another way is to observe that the irreducible representations of $\mathscr{A}(n)$ parametrised by (t,ω) $(t>0)$ remain inequivalent for $\omega = exp\left(\frac{2\pi\sqrt{-1}j}{n}\right)$, $j=0,1,\dots\frac{n}{2}-1$ on restriction to $\overrightarrow{\mathscr{A}(n)}$. This is because $\upsilon_t^2 = \omega^2 e_t$ in that representation. Then adding the sums of squares of the dimensions one gets the number of oriented diagrams by 1.20. □

Finally we make some remarks about generators and relations. As we saw in the introduction, if we put $f_i = u^i e_{n-2} u^{-i}$ (and $F_i = u^r E(n-2;n-2)u^{-i}$) for $i=1,2,\dots,n$, the f_i's satisfy $f_i^2 = f_i$, $f_i f_{i\pm1} f_i = \delta^{-\frac{1}{2}} f_i$ so that if $g_i = qf_i - (1-f_i)$ (for $q+q^{-1}+2=\delta^2$), the map $T_i \mapsto g_i$, $\rho \mapsto u$ gives a homomorphism from the affine Hecke algebra of type A_n with parameter q onto the diagram algebra $A(n, 2+q+q^{-1})$. Thus in particular we have constructed some very explicit irreducible representations of the affine Hecke algebra, for certain values of q.

One reason, besides subfactors, for looking at oriented diagrams in the even case is that they allow us to determine the subalgebra generated by $f_1, f_2, \dots, f_n$ (or $g_1, \dots, g_n$).

LEMMA 2.8. *If* n *is even the following three algebras are equal (even if* $A(n,\delta)$ *is not semisimple).*

(i) *The subalgebra of* $A(n)$ *generated by* $f_1, f_2, \dots, f_n$.

(ii) *The two-sided ideal generated by* f_1 *in* $A(n)$.

(iii) $\overrightarrow{A(n,n-2)}$.

Proof. The equality of (ii) and (iii) follows from a special (oriented) case of (iv) of 2.2.

The algebra of (iii) contains the f_i's by definition. That (iii) implies (i) will follow if we can show that any element of $\overrightarrow{\cup_{t<n} \mathscr{A}(n, n; t)}$ is expressible as a product of F_i's. That this is true for diagrams having a straight through-string is a well known fact about the Temperley-Lieb algebra. But if D is an oriented diagram with less than n through-strings, either D has zero through-string and we are in the Temperley-Lieb situation, or $D \circ u^k$ has a straight through-string for some even k. Thus Du^k is a word on the F_i's and it suffices to show that $F_i u^2$ is a word on the F_i's for all i. It follows from a picture that $F_i u^{-2} = F_i F_{i+1} \dots F_n F_1 F_2 \dots F_{i-2}$. □

Remark 2.9. We leave it to the reader to show that Lemma 2.8 is true without the →'s if n is odd.

Remark 2.10. It follows from 2.8 that the elements v_t are in the algebra generated by the F_i's for $t < n$. We record the expression

$$v_{n-2}^2 = F_n \circ F_1 \circ F_2 \circ \cdots \circ F_n .$$

Thus rotations are unavoidable even if one is only interested in the structure of the algebra generated by the F_i's.

3. THE BRAUER REPRESENTATION

So far we have begged the important question of when the algebra $A(n, \delta)$ is semisimple. We do not have a complete answer for this but we shall show that it is semisimple whenever δ is an integer $\geqslant 3$, (and that $A(n, -2)$ is not semisimple for $n \geqslant 3$) by using a representation onto a C^*-algebra which we will show to be faithful for such δ. That the representation is faithful for n fixed and large integral (hence any large) δ is rather easy.

Definition 3.1. Let V be a vector space of dimension k and basis $w_1, w_2, \dots, w_k$. If the diagram $D \in D(n, n)$ has n connecting edges called ε, define $\beta(D) \in \mathrm{End}(\otimes^n V)$ by the matrix (with respect to the basis $\{w_{a_1} \otimes w_{a_2} \otimes \cdots \otimes w_{a_n} \mid a_i = 1, 2, \dots k\}$ of $\otimes^n V$)

$$\beta(D)_{a_1 a_2 \dots a_n}^{a_{n+1} \dots a_{2n}} = \prod_{\varepsilon} \delta(a_{s(\varepsilon)}, a_{b(\varepsilon)})$$

where $s(\varepsilon)$, $b(\varepsilon)$ are the two ends of the edge ε, labelled from 1 to $2n$, and, just in this formula, δ is the Kronecker δ.

LEMMA 3.2. *$D \mapsto \beta(D)$ defines a homomorphism of $B(n, k)$ (hence $A(n, k)$) onto a C^*-subalgebra of* $\mathrm{End}(\otimes^n V)$.

Proof. This is just the orthogonal case of [B]. The C^*-structure is that for which V is a Hilbert space with orthonormal basis $\{w_i\}$, and it is clear that the adjoint of D is just D read backwards.

Remark 3.3. Since finite-dimensional C^*-algebras are semisimple, this proves that $\beta(B(n,k))$ is always semisimple. Further note that $\beta(A(n,k))$ is also a C^*-algebra.

THEOREM 3.4. *For $k \geqslant 2$, β restricted to $TL(n,k)$ is faithful for all n.*

Proof. The normalized trace on $\operatorname{End}(\otimes^n V)$ defines a Markov trace on $TL(n,k)$ with Markov parameter k^2. Thus by the calculation of [J] or [GHJ], the structure of $\beta(TL(n,k))$ is known and it has the same dimension as $TL(n,k)$.

THEOREM 3.5. *For $k \geqslant 3$, β restricted to $A(n,k)$ is faithful for all n.*

Proof. Let $x = \sum_{D \in \mathscr{A}(n,n)} \lambda_D D$ ($\lambda_D \in \mathbf{C}$) be such that $\beta(x) = 0$. We have seen that $\mathscr{A}(n,n;0)$ actually consists of planar diagrams so by 3.4 we may suppose that $\lambda_D \neq 0$ for some $D \in \mathscr{A}(n,n;t)$, $t \geqslant 1$. Thus by pre- and post-multiplying x by suitable powers of u, we may assume $\lambda_D \neq 0$ for some D with a straight line joining the inner and outer $*$'s. Now split V as $\mathbf{C}w_1 \oplus w_1^{\perp}$. Since dim $V > 2$, dim $w_1^{\perp} \geqslant 2$. Let P be orthogonal projection from $\otimes^n V$ onto $w_1 \otimes (\otimes^{n-1} w_1^{\perp})$. If D is a diagram with the inner and outer $*$'s not connected, $P\beta(D)P = 0$. Also, the set of diagrams with a straight line between the $*$'s is in obvious bijection with $\mathscr{P}(n-1,n-1)$. Thus $0 = P\beta(x)P = \sum_{D \in \mathscr{P}(n-1,n-1)} \lambda_D P\beta(D)P$ and not all the λ_D's are zero.

But the matrix of $P\beta(D)P$ with respect to the basis

$$\{w_1 \otimes (w_{a_1} \otimes \cdots \otimes w_{a_{n-1}}) \mid a_i = 2, 3, \ldots, k\}$$

is clearly that of "$\beta(D)$" for parameters $k-1$ and $n-1$. By 3.4 we conclude $\sum_{D \in \mathscr{P}(n-1,n-1)} \lambda_D D = 0$, a contradiction. □

COROLLARY 3.6. *$\mathscr{A}(n,k)$ and $\overrightarrow{\mathscr{A}(n,k)}$ are semisimple fo k an integer $\geqslant 3$.*

The question naturally arises of finding those values of δ and n for which $\mathscr{A}(n,\delta)$ is semisimple. We observe that for $\delta = -2$, the algebra $\mathscr{A}(n,\delta)$ is not semisimple for $n > 2$. This is because we may use the Brauer representation corresponding to the symplectic case.

Then $\beta(f_1)$ is represented on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$

$$\text{by the matrix } \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \mathrm{id}$$

using a symplectic basis of $\mathbf{C}^2$, and $\beta(u)$ is the obvious cyclic permutation on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$. But then $2 - \beta(f_1)$ is the transposition on $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ exchanging the first two copies of $\mathbf{C}^2$. Thus the image of $\mathscr{A}(n, \delta)$ is the same as that of the group algebra of the symmetric group.

4. THE CYLINDRICAL TRACE

There is a natural trace functional tr on $A(n, \delta)$ defined by $\mathrm{tr}(D) = \delta^{n(D)}$, $n(D)$ being the number of closed loops formed on the cylinder if the inside and outside boundaries of the annulus are identified. We will call this trace the cylindrical trace.

Note 4.1. This trace exists in fact on the whole Brauer algebra — it could be defined in terms of partitions as $\mathrm{tr}(D) = \delta^{n(D)}$ where $n(D)$ is the number of equivalence classes for the equivalence relation generated by D itself and the relation which identifies each point on the top with the corresponding point on the bottom.

Note 4.2. One has the relation $n(D_1 \circ D_2) = n(D_2 \circ D_1)$ so one might try to define a more general trace by replacing δ by an arbitrary complex number. But $n(\alpha, \beta) \neq n(\beta, \alpha)$ in general so one is forced to choose δ.

If δ is a value for which $A(n, \delta)$ is semisimple we know that $A(n, \delta)$ is a direct sum of matrix algebras, so our cylindrical trace is determined by its value on a minimal idempotent in each matrix algebra summand. We will calculate these "weights" of the trace. In order to do this we will need detailed information on the multiplicities of u in each irreducible representation of $A(n,\delta)$.

Definition 4.3. For $n \geqslant t > 0$ the group $\mathbf{Z}/n\mathbf{Z} \times \mathbf{Z}/t\mathbf{Z} (= \{(a, b) \mid a = 0, \ldots, n-1; b = 0, \ldots t-1\}$ acts by linear transformations on $\mathscr{A}(t, n; t)$ by $(a, b)(D) = u^a \circ D \circ u^b$. (The u's on the left and right in this formula are of course different if $n \neq t$.) Let $F_{n,t}(a, b)$ be the number of fixed points for (a, b). Let $F_n(a)$ be the number of fixed points for the action $D \mapsto D \circ u^a$ of $a \in \mathbf{Z}/n\mathbf{Z}$ on $\mathscr{A}(n, 0)$.

LEMMA 4.4. *The multiplicity of an n-th root of unity* η *as an eigenvalue of* u *in the representation* $\pi_{t,\omega}$ $(\omega^t = 1)$ *is* $\frac{1}{nt}\sum_{a=0}^{n-1}\sum_{b=0}^{t-1}\eta^{-a}\omega^{-b}F_{n,t}(a,b)$ *for* $t > 0$ *and* $\frac{1}{n}\sum_{a=0}^{n-1}\eta^{a}F_n(a)$ *for* $t = 0$.

Proof. From the definition of $\pi_{t,\omega}$ it is clear that the multiplicity is trace $\left(p_\omega\left(\frac{1}{n}\sum_{a=0}^{n-1}\eta^{-a}u^a\right)\right)$. □

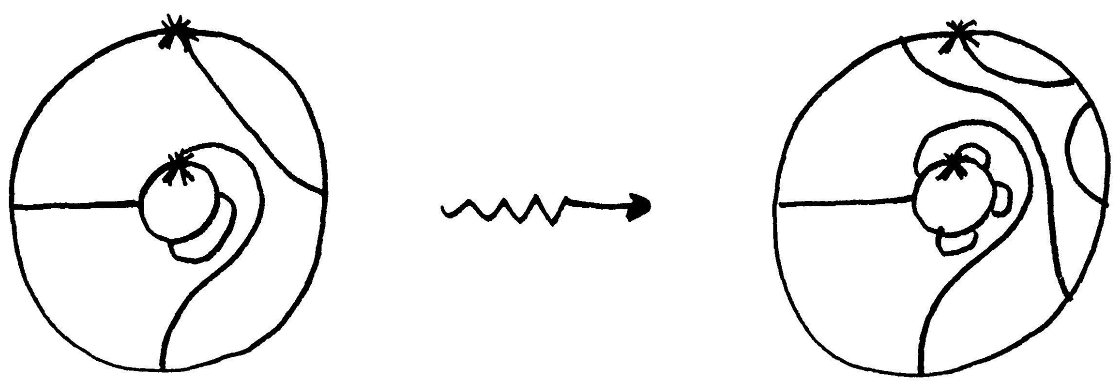
Definition 4.5. For each t-th root of unity ω let $\mathscr{M}(\omega, n)$ (or $\mathscr{M}_{t,n}(\omega, n)$ if it is necessary to specify that ω is indeed a t-th root of unity and not some other) be the cylindrical trace of a minimal projection in the simple summand of $A(n, \delta)$ corresponding to $\pi_{t,\omega}$. To determine $\mathscr{M}(\omega, n)$ we will use the following easy result.

LEMMA 4.6. *For each* $0 \leqslant r < n, r + n$ *even, there is an algebra isomorphism* $\varphi: A(r, \delta) \to e_r A(n, \delta) e_r$ *such that*

(1) $\operatorname{tr}(\varphi(x)) = \operatorname{tr}(x)$, $x \in A(r, \delta)$.

(2) If p *is a minimal projection in the summand of* $A(r, \delta)$ *indexed by* (t, ω), $t \leqslant r$, *then* $\varphi(p)$ *is a minimal projection in the summand of* $A(n, \delta)$ *indexed by* (t, ω).

Proof. Define φ on diagrams by $\varphi(D) = \delta^{\frac{r-n}{2}} D'$, D' differing from D by first inserting $n - r$ interior and exterior points to the right of $*$ and connecting them up in adjacent pairs, very close to the boundary so as to not interfere with the rest of the diagram. Then move $*$ one to the right to ensure that the identity of $A(r, \delta)$ is mapped onto the element we have called e_r. The process of constructing D' from D is illustrated in Figure 4.7.



D D′

FIGURE 4.7

When closed on the cylinder D' will have exactly $\frac{n-r}{2}$ more closed loops than D so $\operatorname{tr}(\phi(x)) = \operatorname{tr}(x)$. The multiplicativity of ϕ also follows from the factor $\delta^{\frac{r-n}{2}}$ in its definition. Injectivity of ϕ is obvious and surjectivity follows by considering a diagram of $E_r \circ D \circ E_r$ for $D \in \mathscr{A}(n, n)$.

Finally, for $t \leqslant r$, $\phi(v_t) = v_t$ (with an obvious abuse of notation) and $\phi(e_t) = e_t$. The summand of $A(r, \delta)$ or $A(n, \delta)$ indexed by (t, ω) is characterized by $v_t = \omega e_t$ (when multiplied by a minimal central idempotent corresponding to the summand). □

We are now in a position to give a formula that determines $\mathcal{M}(\omega, n)$.

THEOREM 4.8. *For* $r < n$, $\mathcal{M}_{r,n}(\omega, n) = \mathcal{M}_{r,r}(\omega, r)$ *and, if* $r = n$,

$$\mathcal{M}(\eta, n) = \frac{1}{n} \sum_{j=1}^{n} \delta^{GCD(j,n)} \eta^j - \sum_{\substack{n > t > 0 \\ t+n \text{ even}}} \sum_{\omega, \omega^t = 1} \mathcal{M}(\omega, t) \left\{ \frac{1}{nt} \sum_{a=0}^{n-1} \sum_{b=0}^{t-1} \eta^{-a} \omega^{-b} F_{n,t}(a, b) \right\} - \frac{1}{n} \sum_{a=0}^{n-1} \eta^{-a} F_n(a) .$$

Proof. Since the ϕ of Lemma 4.6 is surjective, a minimal idempotent in $A(r, \delta)$ is minimal in $A(n, \delta)$ for $r < n$, so by 4.6 we are reduced to the case $r = n$. If we fix an n-th root of unity η, the trace we are trying to calculate is $\mathrm{tr}(P \frac{1}{n} \sum_{j=1}^{n} \eta^j u^j)$ where $(1 - P)$ is the central idempotent of $A(n, \delta)$ corresponding to all matrix summands indexed by (t, ω) with $t < n$. Since the trace of u^j itself is clearly $\delta^{GCD(j,n)}$ one has

$$\mathcal{M}(\eta, n) + \mathrm{tr}\left((1 - P) \frac{1}{N} \sum_{j=1}^{n} \eta^{-j} u^j\right) = \frac{1}{n} \sum_{j=1}^{n} \delta^{GCD(j,n)} \eta^{-j} .$$

Writing $(1 - P) A(n, \delta) (1 - P)$ as a sum of matrix algebras we get the result by 4.4. □

Thus we only need to determine $F(a, b)$ and $F_n(a)$.

THEOREM 4.9. *If* $a = 0, 1, \ldots, n-1$, $b = 0, 1, \ldots, t-1$ $(n \geqslant t,\ t \neq 0)$, *let* $x = GCD(a, n)$, $y = GCD(b, t)$, *then*

$$\text{(a)} \quad F_{n,t}(a, b) = \begin{cases} 0 & \text{if } \frac{a}{x} \neq \frac{b}{y} \text{ or } \frac{b}{x} \neq \frac{t}{y} \text{ or } x \neq y \bmod 2 \\ t \binom{x}{\frac{x-y}{2}} & \text{otherwise (and } a, b \neq 0) \\ 0 & \text{if } a \text{ or } b = 0, \text{ not both, or } n + t \text{ odd} \\ t \binom{n}{\frac{n-t}{2}} & \text{if } a = b = 0 , \end{cases}$$

$$\text{(b)}\quad F_n(a) = \begin{cases} \frac{2m+1}{m+1}\binom{2m}{m}\left(=\binom{2m+1}{m}\right) & \textit{if } n=4m+2 \textit{ and } a=2m+1 \\ \binom{x}{\frac{x}{2}} & \textit{if } x \textit{ is even} \\ \frac{1}{n/2+1}\binom{n}{n/2} & \textit{if } a=0 \textit{ and } n \textit{ is even} \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. Let us prove (b) first as the method is the same for (a) but (b) is simpler.

In the case $n = 4m + 2$, we first claim that for a fixed diagram some point on the boundary must be joined to the point diametrically opposite. This is easy by induction — it is trivial for $n = 2$, and if $n > 2$, just choose two boundary points connected to each other. Either they are diametrically opposite each other and we are done, or the disc is divided into three regions as in Figure 4.10.

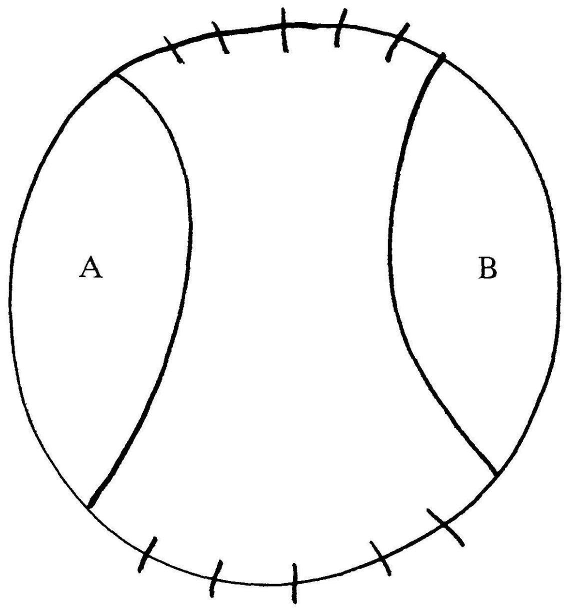


FIGURE 4.10

The boundary points inside A (hence B) are even in number so the number of marked boundary points in the diagram is congruent to 2 mod 4. But the original 180° rotation acts by a 180° rotation on these points so we are done by induction.

Once we know that some point is connected to a diametrically opposite point, the whole diagram, since it is fixed by the rotation of 180°, is determined by the configuration in one half. There are $\frac{1}{m+1}\binom{2m}{m}$ such configurations, and the diameter can be chosen in $2m + 1$ ways.

Now suppose $x = GCD(n, a)$ is even. Then the n boundary points may be divided up into n/x fundamental domains, each consisting of x consecutive points on the boundary. The x points in a fundamental domain can be divided into ones connected to points within the domain and ones connected to points in other fundamental domains. Moreover the constraint of planarity clearly implies that if a point is connected to a point in another fundamental domain, that other domain must be adjacent to it. Thus we may speak of clockwise and anticlockwise points and obviously, since the diagram is fixed, there are the same number of clockwise as anticlockwise points for each domain. We see that the whole diagram is completely determined by a single configuration as in Figure 4.11.

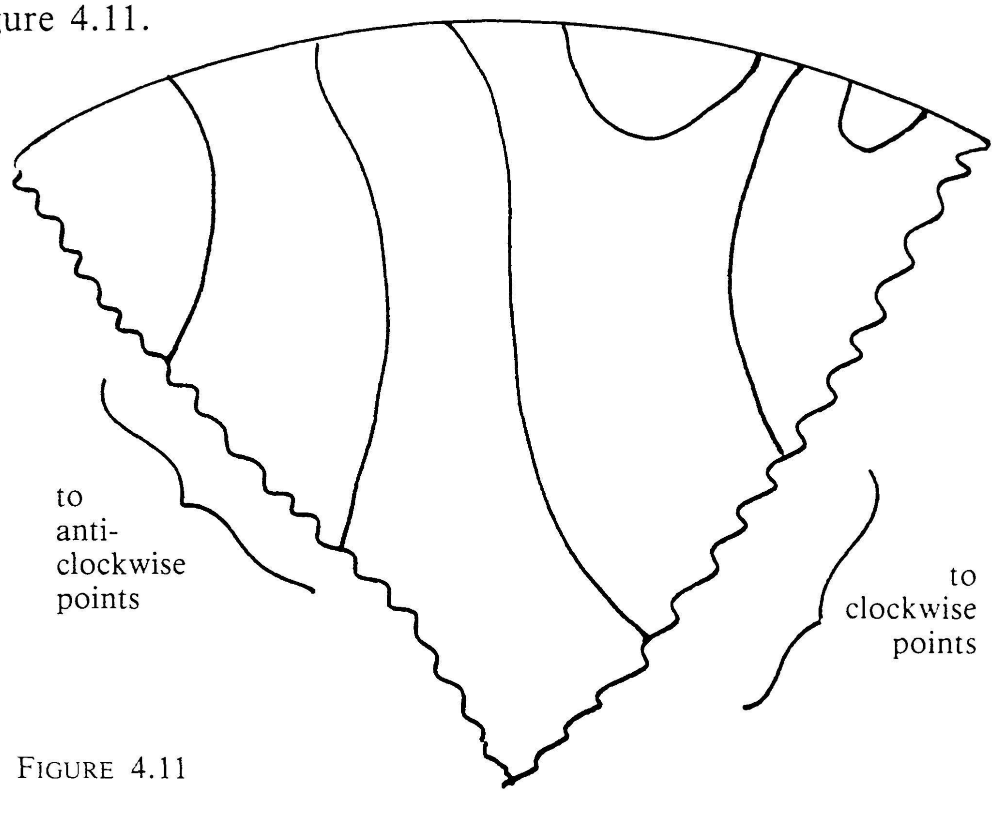


FIGURE 4.11

Also any such configuration determines a fixed point. Straightening out the wavy radii into a single straight line we see that these configurations are in bijection with $\cup_{i=0}^{x/2} \mathscr{P}(x, 2i; 2i)$ which has order $\binom{x}{\frac{x}{2}}$ by Lemma 1.12.

Finally for part (b), if x is odd, there would be an odd number of points in a fundamental domain, which is clearly impossible by the above argument.

Proof of (a). As in the proof of part (b), divide the n outside points into n/x "fundamental domains" for the rotation of a units on the outside circle. Each of the x points in a domain is then of one of four kinds: a "through-point" — attached to the inner circle; a clockwise point — attached to the adjacent domain in clockwise order; an anticlockwise point — similarly; or an internal point — attached to another point in the domain.

The whole system of connections can then be extended to all the outer points by rotating the fundamental domain by powers of the rotation of a units. The through-points can then be connected to the inside points in any of t ways which accounts for the factor of "t" in the formula. That any fixed diagram must look like this follows by arguing only on the outside points. The diagram will then be fixed by (a, b) if and only if the rotation through a points on the outside effects a rotation of b points when restricted to the through-points.

Now suppose there are r through-points per fundamental domain. Obviously $r \cdot \frac{n}{x} = t$, and the rotation of a effects a rotation of $\frac{ra}{x}$ on the through-points. Thus we must have $\frac{ra}{x} = b$, $r\frac{n}{x} = t$. Moreover the through-points in a fundamental domain must be connected to inner points in a fundamental domain for the rotation of b, so $y = r$. So the conditions $\frac{a}{x} = \frac{b}{y}$ and $\frac{n}{x} = \frac{t}{y}$ are necessary for a fixed point. The equality of x and y mod 2 follows from the fact that there have to be as many clockwise points as anticlockwise (as in part (b)) and the number of internal points is necessarily even.

Finally, if all the conditions are satisfied, any configuration as below can be extended in t ways to a fixed point for (a, b).

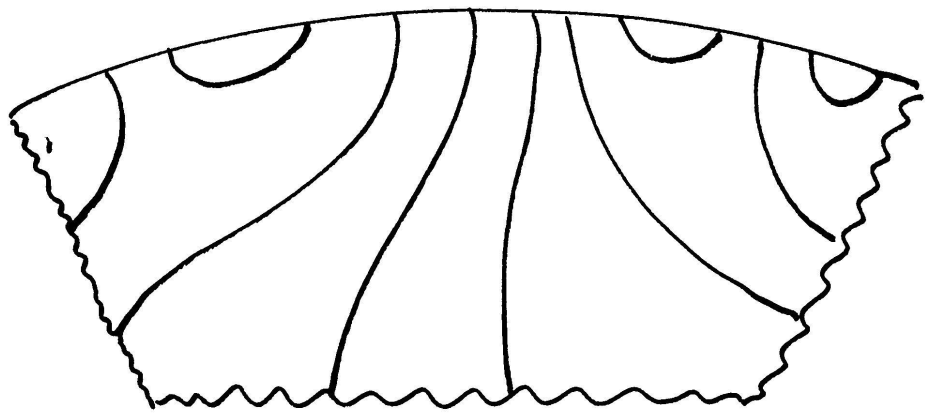


FIGURE 4.12: Two clockwise, two through and six internal points.

As in part (b), make the wavy line one straight line and we see there are $\binom{x}{\frac{x-y}{2}}$ such configurations by Lemma 1.12. □

Given the apparently erratic nature of $F(a, b)$, the elegance of the final formula for $\mathcal{M}(\eta, n)$ seems to us quite remarkable. It will be most transparent if we use the Fourier transform. These are characters rather than multiplicities.

Definition 4.13. For $r = 0, 1, \ldots, n-1$, let $M(r, n) = \sum_\eta \eta^r \mathcal{M}(r, n)$, the sum being taken over all n-th roots of unity η.

The following result was first obtained on a computer by S. Eliahou.

THEOREM 4.14. *If* $T_n(x)$ *is the usual Tchebychev polynomial,* $T_n(\cos\theta) = \cos n\theta$, *then we have*

$$M(0,2) = \delta^2 - 1 = 2T_2\left(\frac{\delta}{2}\right) + 1$$

$$M(1,2) = \delta - 1 = 2T_1\left(\frac{\delta}{2}\right) - 1$$

and for $n > 2$,

$$M(r,n) = 2T_{GCD(n,r)}\left(\tfrac{\delta}{2}\right) .$$

Proof. Let us first obtain the recursive formula for $M(r,n)$ from Theorem 4.8:

$$\sum_{\eta:\eta^n=1} \mathcal{M}(\eta,n)\eta^r = \delta^{GCD(r,n)} - \left\{ \sum_{\substack{t>0 \\ t+n \text{ even}}}^{n-2} \ \sum_{\omega:\omega^t=1} \mathcal{M}(\omega,t)\, \frac{1}{t} \sum_{b=0}^{t-1} \omega^{-b} F_{n,t}(r,b) \right\}$$
$$- F_n(r)$$
$$= \delta^{GCD(r,n)} - \sum_{\substack{t>0 \\ t+n \text{ even}}}^{n-2} \frac{1}{t} \sum_{b=0}^{t-1} M(t-b,t) F_{n,t}(r,b) - F_n(r) ,$$

where the last term is only present if n is even.

We must show that the function defined in the statement of the theorem, call it $\mu(r,n)$, satisfies this recursion equation. Note first that if we set $P_n(x) = 2T_n\left(\frac{x}{2}\right)$ then (see [Lu]):

$$P_n(x) = x^n - \sum_{\substack{0 \leq i < n \\ i+n \text{ even}}} \binom{n}{\frac{n-i}{2}} P_i(x) .$$

The case $r = 0$ is now rather easy: For $n = 1$, $M(0,1) = \delta - F_1(0) = \delta$ since there are no diagrams with one boundary point. Also $\mu(0,1) = P_1(\delta) = \delta$. For $n = 2$, $M(0,2) = \delta^2 - F_2(0) = \delta^2 - 1 = \mu(0,1)$. For $n > 2$ we have (first for n even).

$$\mu(0,n) = P_n(\delta) = x^n - \sum_{\substack{n>t>2 \\ t \text{ even}}} \binom{n}{\frac{n-t}{2}} \mu(0,t) + \binom{n}{\frac{n-2}{2}} (\delta^2 - 2) + \binom{n}{\frac{n}{2}}$$

$$= x^n - \sum_{\substack{n>t>2 \\ t \text{ even}}} \frac{1}{t} F_{n,t}(0,0)\mu(0,t) + \tfrac{1}{2} F_{n,2}(0,0)(\delta^2 - 1) + \binom{n}{\frac{n}{2}} - \binom{n}{\frac{n-1}{2}}$$

but since $F_{n,t}(0, b) = 0$ for $b \neq 0$, we get

$$\mu(0, n) = x^n - \sum_{\substack{n > t > 0 \\ t \text{ even}}} \frac{1}{t} \sum_{b=0}^{t-1} F_{n,t}(0, b)\mu(t-b, t) - F_n(0) .$$

The case where n is odd is even easier.

Now consider the case where r is arbitrary. Since $GCD(b, t) = GCD(t-b, t)$ we must show

$$\mu(r, n) = \delta^{GCD(r,n)} \sum_{\substack{t > 0 \\ t+n \text{ even}}}^{n-2} \frac{1}{t} \sum_{b=0}^{t-1} \mu(b, t) F_{n,t}(r, b) - F_n(r) .$$

Let us find all pairs (t, b), $0 < b \leqslant t-1$, $0 < t < n$, for which $F_{n,t}(r, b) \neq 0$. Let $g = GCD(r, n)$. Then from 4.6 we must have $t = \frac{\alpha n}{g}$, $b = \frac{\alpha r}{g}$, $\alpha + g$ even and $\alpha < g$, for $\alpha = GCD(r, n)$. On the other hand, if we are given an α with $\alpha + g$ even and $0 < \alpha < g$, then $GCD\left(\frac{\alpha n}{g}, \frac{\alpha r}{g}\right) = \alpha$ so if we put $t = \alpha n/g$, $b = \alpha r/g$, $0 < t < n$, $0 < b < t$ and $GCD(t, b) = \alpha$. Thus since $F_{n,t}(r, 0) = 0$ for $r \neq 0$, the equation to check becomes

$$\mu(r, n) = \delta^g - \sum_{\substack{\alpha > 0 \\ \alpha+g \text{ even}}}^{g-2} \mu\left(\frac{\alpha r}{g}, \frac{\alpha n}{g}\right) \binom{g}{\frac{g-\alpha}{\alpha}} - F_n(r) .$$

On the other hand,

$$P_g(\delta) = \delta^g - \sum_{\substack{\alpha > 0 \\ \alpha+g \text{ even}}}^{g-2} P_\alpha(\delta) \binom{g}{\frac{g-\alpha}{2}} - \binom{g}{\frac{g}{2}},$$

where the last term is present in the even case (for g) only.

Thus we are done if g is odd since then $F_n(r) = 0$ and there is no difference between these recursion relations. The sum in the expression for μ can only contain $\mu(1, 2)$ if $n = 2r$ so $g = r$ and $\alpha = 1$. In this case g is odd, so we are done in the case g even since then $F_n(r) = \binom{g}{\frac{g}{2}}$ *and* $\mu(a, b) = \mathscr{P}_{GCD(a,b)}$ by definition for all terms in the sum for μ. Finally there is the case $n = 2r$ $g(= r)$, odd. Then all the terms in the recursions are the same except the last two — for $\mu(r, n)$ we have

$$\mu(1, 2)\binom{g}{\frac{g-1}{2}} + \binom{g}{\frac{g-1}{2}} = (P_1 - 1)\binom{g}{\frac{g-1}{2}} + \binom{g}{\frac{g-1}{2}} = P_1\binom{g}{\frac{g-1}{2}}$$

which is the same as the last term in the formula for $P_g(\delta)$. □

COROLLARY 4.15. *The traces of minimal idempotents in the matrix algebra summand corresponding to* (ω, t), $\mathcal{M}(\omega, t)$, *are given by:*

$$\text{For } t = 1, \quad \mathcal{M}(1,1) = 1$$

$$\text{For } t = 2, \quad \mathcal{M}(1,2) = \frac{\delta^2 + \delta - 2}{2}, \qquad \mathcal{M}(-1,2) = \frac{\delta^2 - \delta}{2}$$

$$\text{For } t > 2, \quad \mathcal{M}(\omega, t) = \frac{1}{t} \sum_{r=0}^{t-1} 2T_{GCD(r,n)}\left(\frac{\delta}{2}\right)\omega^r$$

$$= \frac{2}{t} \sum_{d \mid t} \left(\sum_{\substack{k : GCD(n,k) = d \\ k \leqslant n}} \omega^k \right) T_k\left(\frac{\delta}{2}\right).$$

Proof. Just invert the Fourier transform. □

COROLLARY 4.16. *The multiplicity of the representation* $\pi_{t,\omega}$ *of* $A(n,k)$ *in the Brauer representation* β *(§3) is* $\mathcal{M}(\omega, t)(k)$, *for* $k \geqslant 3$. *(So* $\mathcal{M}(\omega, k) > 0$ *for* $k \geqslant 3$.*)*

Proof. For $k \geqslant 3$ the algebra is semisimple and the trace induced by the usual trace of $\operatorname{End}(\otimes^n V)$ is the cylindrical trace, with parameter $\delta = k$. □

If we look at the oriented subalgebra $\overrightarrow{A}(n, \delta)$ (with n even), the irreducible representations are parametrised by even t's and the first $t/2$ t-th roots of unity ω. Obviously $\mathcal{M}(\omega, t) = \mathcal{M}(\bar{\omega}, t)$ since $GCD(r,n) = GCD(n-r, n)$. Let $\overrightarrow{\mathcal{M}}(\omega, t)$ denote the cylindrical trace of a minimal idempotent in the summand corresponding to $\pi(t, \omega)$.

COROLLARY 4.17. $\vec{\mathcal{M}}(\omega, t) = \frac{4}{t} \sum_{r=0}^{t-1} T_{GCD(2,n)}\left(\frac{\delta}{2}\right)\omega^r$, *for* $n > 2$.

Proof. On restriction to $\vec{A}(n, \delta)$ the representations of A parametrised by ω and $-\omega$ become equivalent.

COROLLARY 4.18. *The Brauer representation* β *is not faithful for* $k = 2$ *and* $n \geqslant 3$.

Proof. $T_n(1) = 1$ so for $\omega \neq 1$, $\mathcal{M}(\omega, t) = 0$, and this is sufficient to imply that the matrix algebra corresponding to ω is in the kernel of β.

APPENDIX 1

Table A.1.1

The dimensions of the irreducible representations of $A(n, \delta)$
(grouped according to the number of through strings)

dimension					
	1				
1	1 1				
	3	1 1 1			
2	4 4	1 1 1 1			
	10	5 5 5	1 1 1 1 1		
5	15 15	6 6 6 6	1 1 1 1 1 1		
	35	21 21 21	7 7 7 7 7	1 1 1 1 1 1 1	
14	56 56	28 28 28 28	8 8 8 8 8 8	1 1 1 1 1 1 1 1	
	126	84 84 84	36 36 36 36 36	9 9 9 9 9 9 9	1 1 1 1 1 1 1 1 1

Table A.1.2

The multiplicities in the Brauer representation. [1] [2]

n	η	**multiplicity**
1	1	1
2	1	$k^2 + k - 2$
	-1	$k^2 - k$
3	1	$k^3 - k$
	$\exp(2i\pi/3)$	$k^3 - 4k$
4	1	$k^4 - 3k^2 + 2k$
	i	$k^4 - 5k^2 + 4$
	-1	$k^4 - 3k^2 - 2k$
5	1	$k^5 - 5k^3 + 9k$
	$\exp(2i\pi/5)$	$k^5 - 5k^3 + 4k$
6	1	$k^6 - 6k^4 + k^3 + 11k^2 - k - 6$
	$\exp(i\pi/3)$	$k^6 - 6k^4 - k^3 + 8k^2 + 4k$
	$\exp(2i\pi/3)$	$k^6 - 6k^4 - k^3 + 11k^2 + k - 6$
7	1	$k^7 - 7k^5 + 14k^3 - k$
	$\exp(2i\pi/7)$	$k^7 - 7k^5 + 14k^3 - 8k$
8	1	$k^8 - 8k^6 + 21k^4 - 18k^2 + 4k$
	$\exp(i\pi/4)$	$k^8 - 8k^6 + 19k^4 - 12k^2$
	i	$k^8 - 8k^6 + 21k^4 - 22k^2 + 8$
	-1	$k^8 - 8k^6 + 21k^4 - 18k^2 + 4k$
9	1	$k^9 - 9k^7 + 27k^5 - 28k^3 + 9k$
	$\exp(2i\pi/9)$	$k^9 - 9k^7 + 27k^5 - 31k^3 + 12k$
	$\exp(2i\pi/3)$	$k^9 - 9k^7 + 27k^5 - 28k^3$

[1]) Since $M_n(t, \omega) = M_t(t, \omega)$ we record just $M_n(n, \eta)$ for η an nth root of unity.

[2]) The table entry gives $nM_n(n, \eta)$ for a root of unity of order d for each divisor d of n.

APPENDIX 2: RESTRICTION TO THE TEMPERLEY-LIEB ALGEBRA

The Temperley-Lieb algebra $P(n, \delta)$ (see §2) is contained (unitally) in $A(n, \delta)$ (indeed in $\overrightarrow{A(n, \delta)}$) by simply connecting the inside $*$ to the outside $*$, which reduces the rest of the annulus to a disc. The structure of $P(n, \delta)$ is very well known, particularly when it is semisimple (see [GHJ], and [GW] in the non-semisimple case). This structure is very easily re-obtained by the method of this paper. We have that there is one irreducible representation of $P(n, \delta)$ for each t, $0 \leqslant t \leqslant n$, $t + n$ even, of dimension $\binom{n}{\frac{n-t}{2}} - \binom{n}{\frac{n-t-2}{2}}$.

Call these representations ψ_t.

THEOREM. *For* $t > 0$,

$$\pi_{t,\omega|P(n,\delta)} = \bigoplus_{\substack{t \leqslant k \leqslant n \\ k + t \text{ even}}} \psi_k$$

and when $t = 0$,

$$\pi_{0|P(n,\delta)} = \psi_0$$

(when both algebras are semisimple).

This is easily proved by induction using Theorem 2.8 and Lemma 4.6. It is reassuring to note that the dimensions add up in an obvious way:

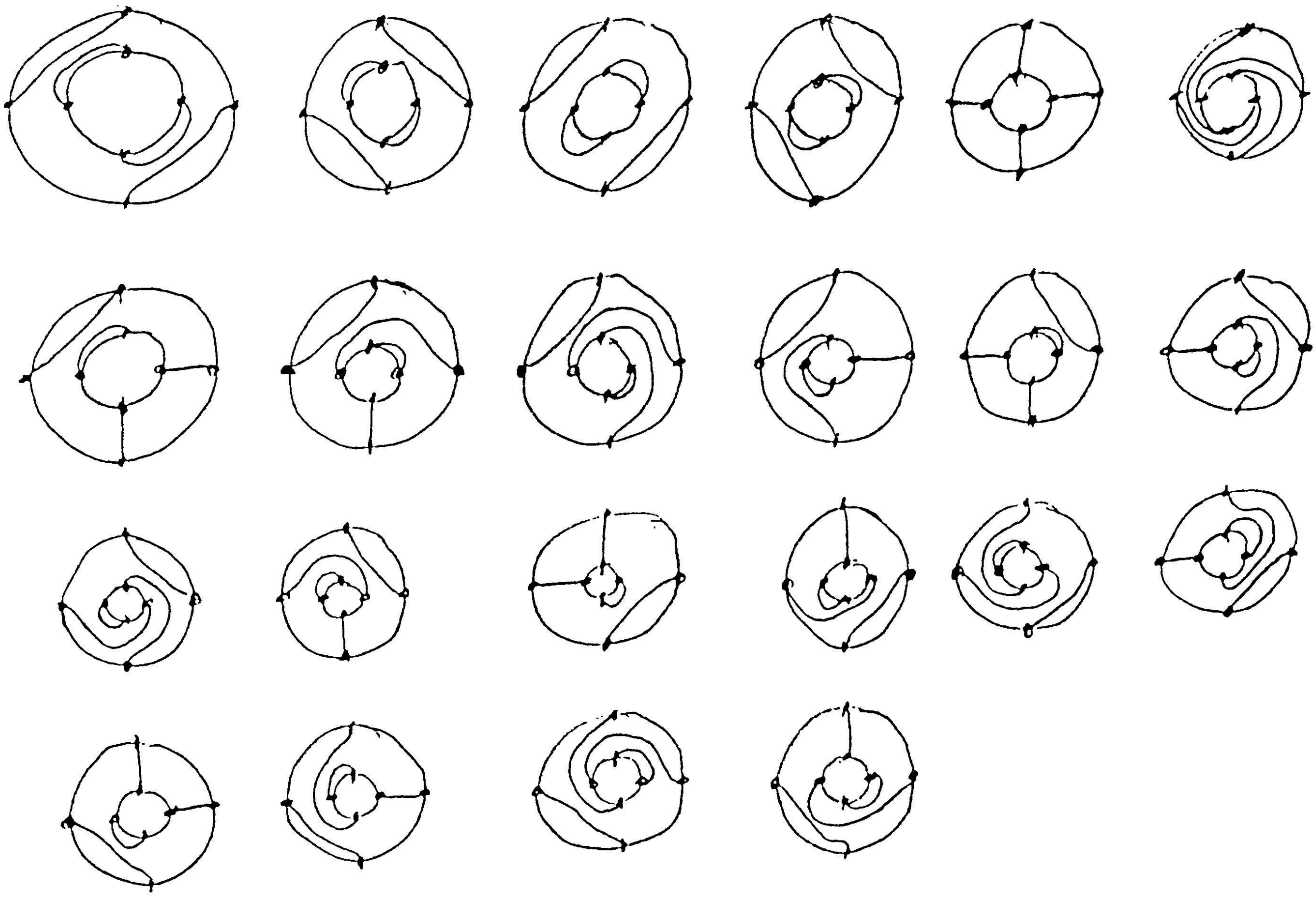
$$\dim(\pi_{t,\omega}) = \binom{n}{\frac{n-t}{2}} = \left\{\binom{n}{\frac{n-t}{2}} - \binom{n}{\frac{n-t}{2} - 1}\right\} + \left\{\binom{n}{\frac{n-t}{2} - 1} - \binom{n}{\frac{n-t}{2} - 2}\right\}$$
$$+ \cdots + \left\{\binom{n}{0} - \binom{n}{-1}\right\}.$$

Similarly one may check that the formulas for the traces of minimal idempotents add up.

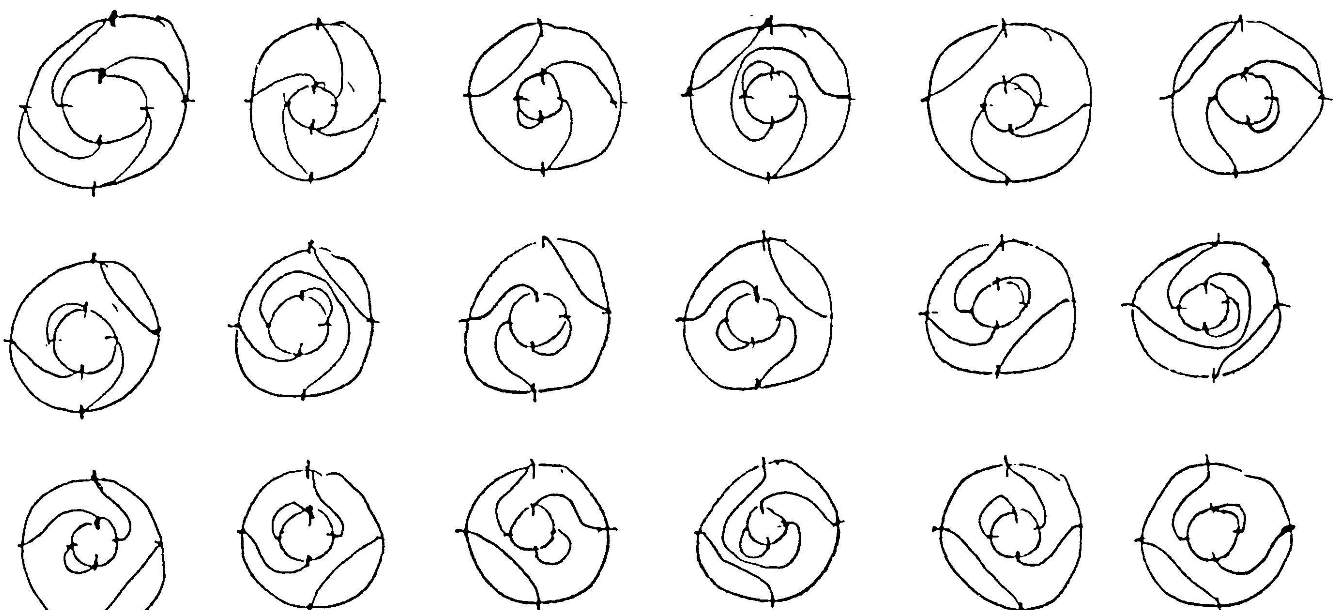
Our first attempt to derive the structure of $A(n, \delta)$ was using the unital inclusion of the Temperley-Lieb algebra. The only stumbling block was in trying to show that the "trivial" representation ψ_n (of dimension 1) is actually contained in $\pi_{t,\omega}$.

APPENDIX 3: The elements of $\mathscr{A}(4, 4)$.

Oriented diagrams:



Unoriented diagrams:



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V.F.R. Jones

Department of Mathematics
University of California
Berkeley, CA 94720