1. COUNTING DIAGRAMS

Definition 1.1. An (a, b) *diagram* D will be a partition of the union of a set of size "a" and a set of size "b" into subsets of size 2 (so $a + b$ is even). If the set of size "a" consists of points on one line and the set of size "b" consists of points on another, the diagram may be represented pictorially as below:

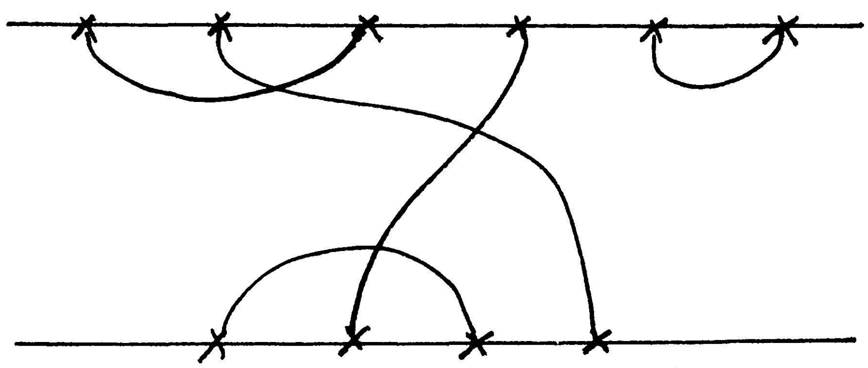


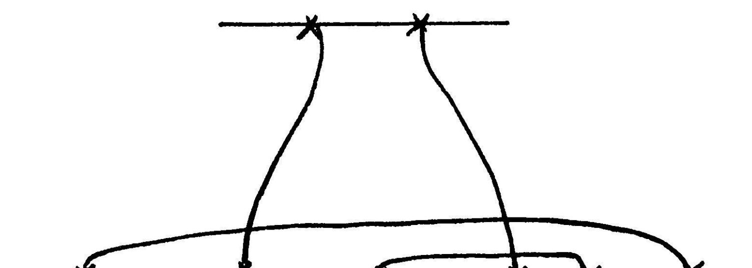
FIGURE 1.2: A (4, 6) diagram α.

The set of all (a, b) diagrams will be denoted $\mathscr{D}(a, b)$.

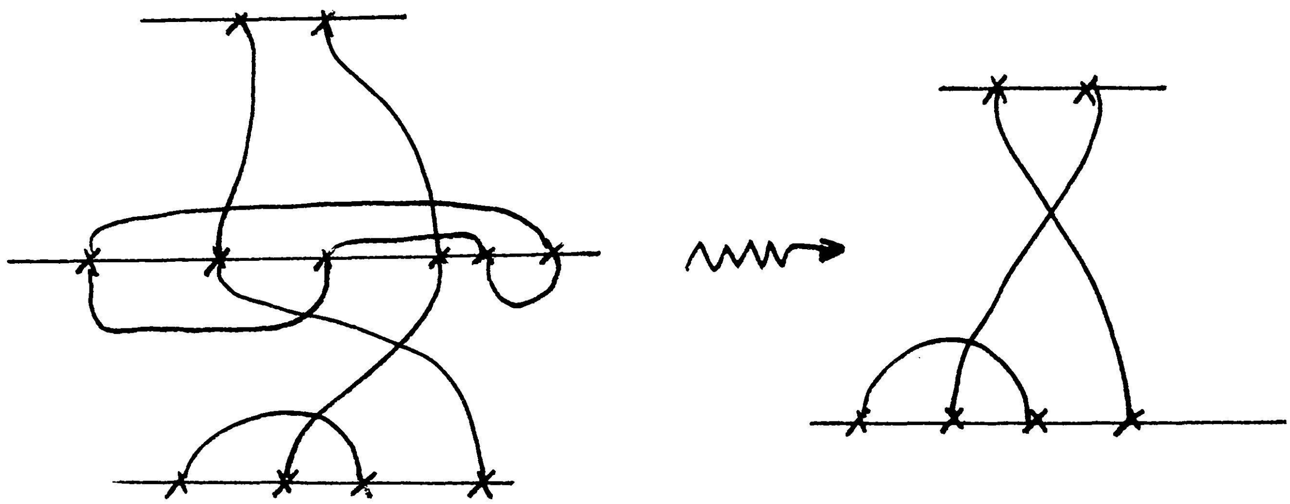
Definition 1.3. If $\alpha \in \mathscr{D}(a, b)$ a *through-string* for the pictorial representation of α is a line going from one of the top points to one of the bottom ones. The number of through strings of α will be written $t(\alpha)$.

If there is some way of identifying sets of a given size (such as if they are points on a line or a circle), we may define an associative rule $(\alpha, \beta) \mapsto \alpha \circ \beta$ allowing one to multiply a $\mathscr{D}(a, b)$ diagram by a $\mathscr{D}(b, c)$ diagram to get a $\mathscr{D}(a, c)$ diagram. No doubt the neatest way to define all this is in terms of categories but we prefer to remain extremely concrete so we define the associative rule pictorially by concatenation of diagrams, as below.

FIGURE 1.4. With α as in Figure 1.1 and $\beta \in \mathscr{D}(6, 2)$ as follows,



$\alpha \circ \beta$ is formed as follows:



When two diagrams α and β are multiplied, a certain number of closed loops are formed which are forgotten in the product $\alpha \circ \beta$. We call $n(\alpha, \beta)$ the number of closed loops.

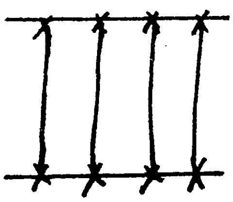
LEMMA 1.5. *If* $\alpha \in \mathscr{D}(a, b)$, $\beta \in \mathscr{D}(b, c)$, $\gamma \in \mathscr{D}(c, d)$,

(i) $n(\alpha, \beta) + n(\alpha \circ \beta, \gamma) = n(\beta, \gamma) + n(\alpha, \beta \circ \gamma)$

(ii) $t(\alpha \circ \beta) \leqslant \min\{t(\alpha), t(\beta)\}$

Proof. (i) Both sides count the number of closed loops in the figure obtained by concatenating the figures of α, β and γ.

(ii) Obvious.

The diagram  in $\mathscr{D}(n, n)$ is obviously an identity for $\circ$ and will be denoted 1.

Definition 1.6. We will say that a diagram $D \in \mathscr{D}(a, b)$ is *planar* if a figure representing the diagram has no crossings and all the connecting lines do not leave the strip in the plane defined by the top and bottom lines.

We will say that D is *annular* if the $a + b$ points are on the inside and outside of an annulus with all connecting lines inside the annulus, and without crossings as in Figure 1.7.

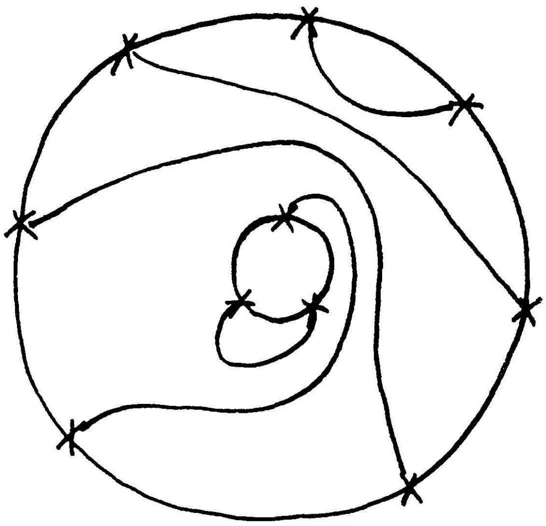


FIGURE 1.7: An annular (3, 7) diagram with 1 through-string.

The set of planar (a, b) diagrams will be denoted $\mathscr{P}(a, b)$ and those having t through-strings $\mathscr{P}(a, b; t)$, and the set of annular (a, b) diagrams $\mathscr{A}(a, b)$ and those having t through-strings $\mathscr{A}(a, b; t)$. It is well known that the set $\mathscr{P}(n, n)$ has order $\operatorname{cat}(n) = \frac{1}{n+1}\binom{2n}{n}$. We shall count $\mathscr{A}(a, b)$.

It will be convenient to fix two points on the inside and outside circle respectively, call them $*$, and suppose they lie on a vertical line thus:

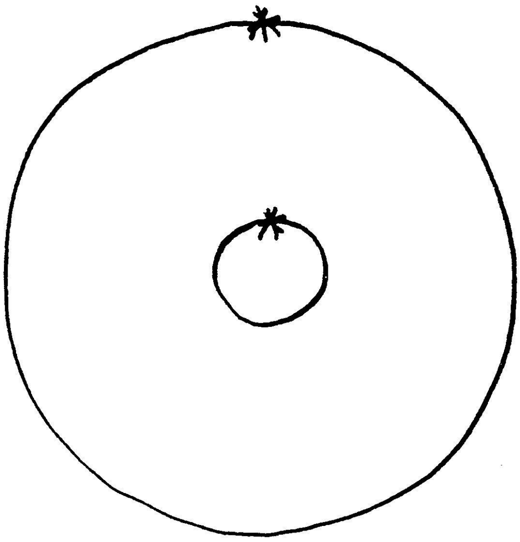


FIGURE 1.8

Because of applications to subfactors we shall be interested in the case $\mathscr{A}(n, n)$, for which we say that a diagram is *oriented* if the curves joining points may be oriented so that, at the boundary points, they point alternately inward and outward, with the top $*$ pointing outward and the bottom $*$ pointing inward. Thus Figure 1.9 is a figure of an oriented diagram.

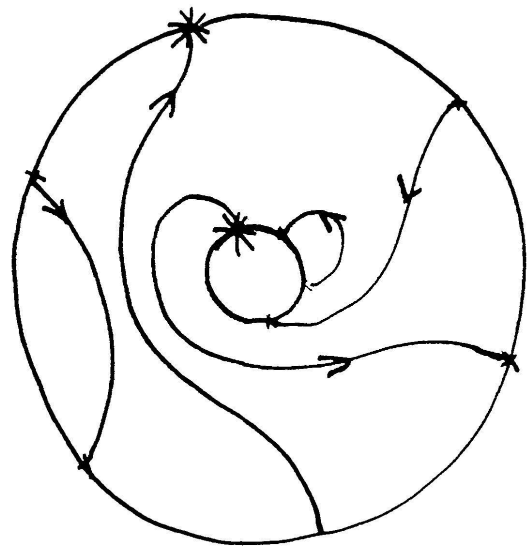


FIGURE 1.9

The set of oriented diagrams will be denoted $\vec{\mathscr{A}}(n, n)$. Note that any planar diagram is oriented. There are 22 oriented diagrams in $\mathscr{A}(4, 4)$ and 40 elements altogether, which we enumerate in appendix 3 as they should be quite useful in understanding the rest of the paper.

The difference between the planar and annular cases is that the curves in the diagram may go round the circle. It is easy to guess that the element we are about to define in $\mathscr{A}(n, n; n)$ will have a significant role to play.

Definition 1.10. The element $u \in \mathscr{A}(n, n; n)$ is the diagram of a cyclic permutation represented by the figure below.

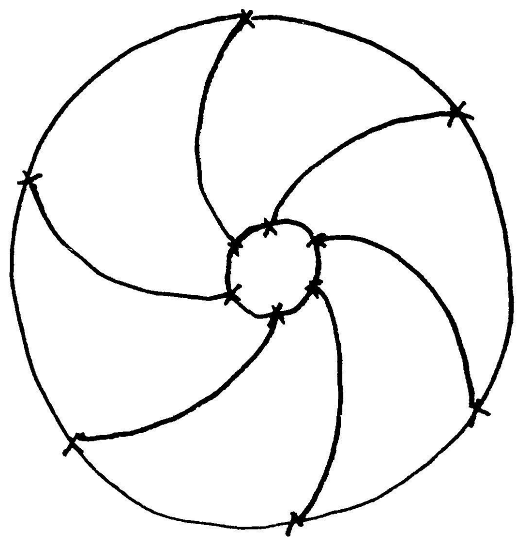


FIGURE 1.11: $u \in \mathscr{A}(6, 6; 6)$

PROPOSITION 1.12. *For the semigroup structure on* $\mathscr{A}(n, n)$ *determined by* o, $u^n = 1$.

Proof. Although the figure represented by u^n has a 360° twist, the corresponding diagram is the identity. □

We now want to count annular diagrams. We begin with a preliminary result which is well known but we include a proof for the convenience of the reader.

LEMMA 1.13. *Let* $\left\{ {n \atop p} \right\} = |\mathscr{P}(n,p;p)|$, *the number of planar* (n,p) *diagrams with* p *through-strings. Then*

$$\left\{ {n \atop p} \right\} = \binom{n}{\frac{n-p}{2}} - \binom{n}{\frac{n-p-2}{2}} = \frac{p+1}{n+1}\binom{n+1}{\frac{n-p}{2}}.$$

Proof. Consider an element of $\mathscr{P}(n+1, p-1; p-1)$. The leftmost of the bottom $n+1$ points, call it x, is either connected to the top or the bottom. If it is connected to the top it must be to the leftmost of those $p-1$ points since there are $p-1$ through-strings. There are thus $\left\{ {n \atop p-2} \right\}$ such. If x is connected to the bottom, one may move x to the top to obtain an element of $\mathscr{P}(n,p)$ as below:

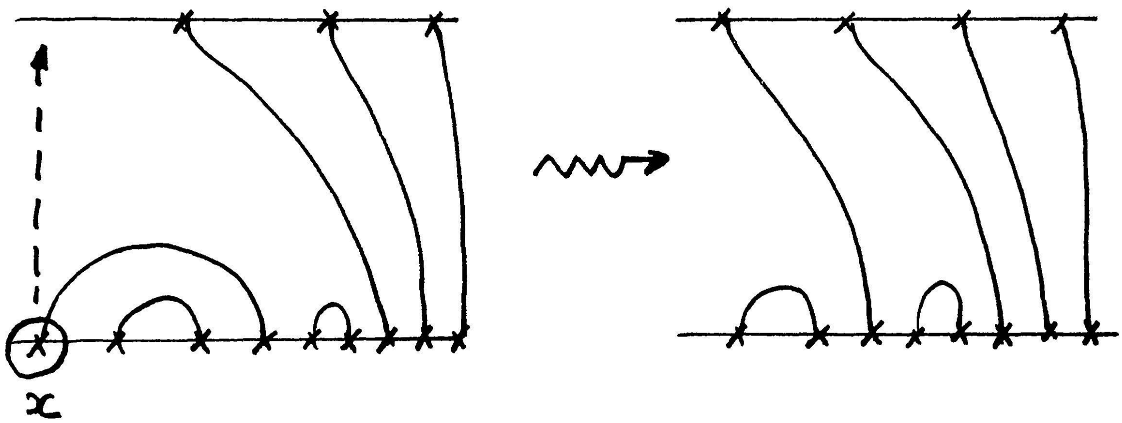


FIGURE 1.14

The process may clearly be reversed so that there are $\left\{ {n \atop p} \right\}$ such and

$$\left\{ {n+1 \atop p-1} \right\} = \left\{ {n \atop p} \right\} + \left\{ {n \atop p-2} \right\}.$$

Putting $\left\langle {n \atop p} \right\rangle = \left\{ {n \atop n-2p} \right\}$ for $p = 0, 1, \ldots \left[\frac{n}{2}\right]$ we see that

$$\left\langle {n+1 \atop p} \right\rangle = \left\langle {n \atop p} \right\rangle + \left\langle {n \atop p-1} \right\rangle .$$

But the same recursion relations and boundary conditions are satisfied by $\binom{n}{p} - \binom{n}{p-1}$. So

$$\left\langle {n \atop p} \right\rangle = \binom{n}{p} - \binom{n}{p-1} \quad \text{and} \quad \left\{ {n \atop p} \right\} = \left\langle {n \atop \frac{n-p}{2}} \right\rangle = \binom{n}{\frac{n-p}{2}} - \binom{n}{\frac{n-p-2}{2}} .$$

Also note $\binom{n}{k} - \binom{n}{k-1} = \dfrac{n-2k+1}{n+1}\binom{n+1}{k}$. □

COROLLARY 1.15. $|\mathscr{A}(n,p;p)| = p\binom{n}{\frac{n-p}{2}}$ *for* $p \neq 0$.

Proof. Fix one of the p outside points. There are n ways to connect it to the inside. Once connected, one may cut the annulus open along that string and one is in the planar situation so

$$|\mathscr{A}(n,p;p)| = n\left\{ {n-1 \atop p-1} \right\} = \frac{np}{n}\binom{n}{\frac{n-p}{2}} = p\binom{n}{\frac{n-p}{2}} . \quad \square$$

COROLLARY 1.16. $|\mathscr{A}(p,q;t)| = t\binom{p}{\frac{p-t}{2}}\binom{q}{\frac{q-t}{2}}$ *for* $t > 0$.

Proof. Given an annular diagram D with $t(D) = t$, one may push all the curves connecting inner (resp. outer) boundary points to lie within a small neighborhood of the inner (resp. outer) boundary circles. Then D may be cut with a third circle, concentric to the others, which meets only the through-strings. Thus we can write $D = D_1 \circ D_2$ with $D_1 \in \mathscr{A}(p,t;t)$, $D_2 \in \mathscr{A}(t,q;t)$. Moreover given the pair D_1, D_2, it is immediate that any other pair E_1, E_2 with $E_1 \circ E_2 = D$ is of the form $E_i = D_1 \circ u^k$, $E_2 = u^{-k} \circ D_2$ for some $k = 1, 2, \ldots, t$ ($u \in \mathscr{A}(t,t)$ as in Definition 1.10). And all t such pairs (E_1, E_2) are clearly distinct. Thus

$$|\mathscr{A}(p,q;t)| = \frac{1}{t}|\mathscr{A}(p,t;t)||\mathscr{A}(t,q;t)| = t\binom{p}{\frac{p-t}{2}}\binom{q}{\frac{q-t}{2}} . \quad \square$$

COROLLARY 1.17.

$$|\mathscr{A}(2p,2q)| = \mathrm{cat}(p)\,\mathrm{cat}(q) + \sum_{i=1}^{\min(p,q)} 2i \binom{2q}{q-i}\binom{2p}{p-i} \quad \textit{and}$$

$$|\mathscr{A}(2p+1,2q+1)| = \sum_{i=0}^{\min(p,q)} (2i+1)\binom{2p+1}{p-i}\binom{2q+1}{q-i}.$$

Proof. The only case not covered explicitly by Corollary 1.16 is the case of zero through-strings. This can only happen if p and q are even, and then the argument of 1.16 shows that the number is $|\mathscr{A}(2p,0)\|\mathscr{A}(2q,0)| = |\mathscr{P}(2p,0)\|\mathscr{P}(2q,0)|$. □

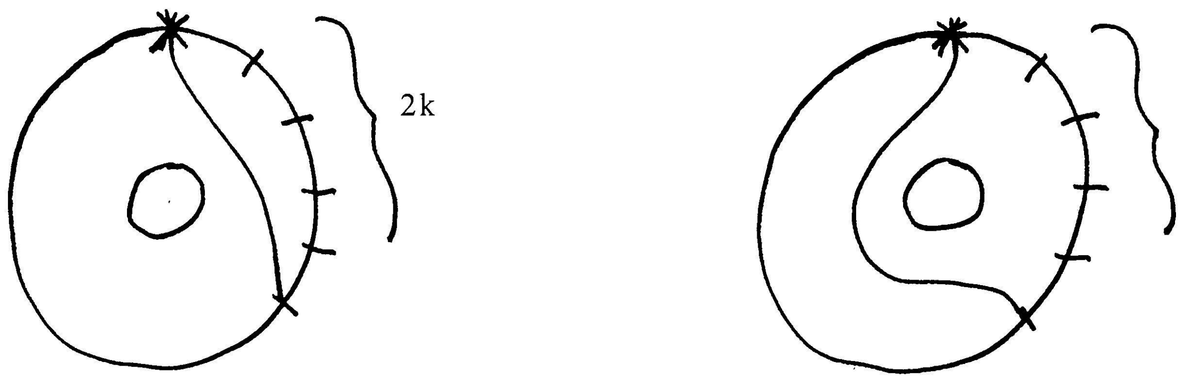
S. Eliahou has calculated $|\mathscr{A}(2p,2q)| = \frac{pq}{p+q}\binom{2p}{p}\binom{2q}{q} + \mathrm{cat}(p)\,\mathrm{cat}(q)$.

It is possible to count the elements in $\mathscr{A}(2p,2q)$ in a completely different way which we now explain. This other way will not be used, but the truth of the resulting binomial identity confirms the calculation.

To save on notation let $d_{p,q} = |\cup_{t>0}\mathscr{A}(2q,2p;t)|$ and let c_k also stand for $\mathrm{cat}(k)$.

First count all the elements of $\mathscr{A}(2q,2p)$ with (the outer) $*$ connected to the inner circle. Clearly $*$ can be connected to $2q$ points and, once connected, there are $\mathrm{cat}(p+q-1)$ ways of completing the diagram. There are thus $2q\,\mathrm{cat}(p+q-1)$ such.

Now assume $*$ is connected to the outside. Then D is of the form (a) or (b) in Figure 1.18:



(a) (b)

FIGURE 1.18

Thus for the $p-1$ possible points to which $*$ is connected we must count the two possibilities (which are distinct since there is at least one through-string), which are clearly $d_{p-k-1,q}c_k$ and $c_{p-k-1}d_{k,q}$. Altogether we get $d_{p,q} = q\,\mathrm{cat}(p+q-1) + c_0 d_{p-1,q} + c_2 d_{p-2,q} + \cdots + c_{p-2} d_{1,q} + d_{1,q} c_{p-2} + \cdots + c_0 d_{p-1,q}$ or

$$d_{p+1,q} = q\,\mathrm{cat}(p+q) + 2\sum_{i=0}^{p-1} c_i d_{p-i,q}$$

To get an explicit formula for $d_{p,q}$, we use generating functions. Let $\mathrm{CAT}(x) = \sum_{n=0}^{\infty} \mathrm{cat}(n) x^n = \frac{1-\sqrt{1-4x}}{2x}$ and $f_q(x) = \sum_{r=0}^{\infty} d_{r+1,q} x^r$. Then

$$f_q(x) = q \sum_{n=0}^{\infty} \mathrm{cat}(q+n) x^n + 2x\mathrm{CAT}(x) f_q(x)$$

so that

$$\begin{aligned} f_q(x) &= q(1-4x)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \mathrm{cat}(q+n) x^n \\ &= q\left(\sum_{r=0}^{\infty} \binom{2r}{r} x^r\right)\left(\sum_{n=0}^{\infty} \frac{1}{n+q+1}\binom{2n+2q}{n+q} x^n\right). \end{aligned}$$

Equating coefficients we see we have proven:

LEMMA 1.19.

$$\begin{aligned} |\mathscr{A}(2p,2q)| - \mathrm{cat}(p)\mathrm{cat}(q) &= 2q \sum_{n=0}^{p-1} \frac{1}{p+q-n}\binom{2n}{n}\binom{2p+2+2q-2n-2}{p+q-n-1} \\ &= \sum_{t=1}^{\min(p,q)} 2t \binom{2p}{p-t}\binom{2q}{q-t} \end{aligned}$$

Finally we observe that oriented diagrams are easily counted from nonoriented ones.

LEMMA 1.20. *If* p *is even and* $u \in \mathscr{A}(p,p)$ *is as in 1.10, then for* $t > 0$, $\alpha \mapsto u\alpha$ *is a bijection between* $\overrightarrow{\mathscr{A}}(p,q;t)$ *and unoriented elements of* $\mathscr{A}(p,q;t)$. *Thus* $|\mathscr{A}(p,q;t)| = 2|\mathscr{A}(p,q;t)|$.

Proof. It suffices to show that, if α is oriented, $u\alpha$ is not, and vice versa. The first assertion is obvious. So if α is not oriented, choose a non-oriented through-string. The same string extended through $u\alpha$ is then oriented. Cutting along that string we are in the planar situation which is necessarily oriented for obvious parity reasons. □