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A PROOF OF MARGULIS' THEOREM ON VALUES OF QUADRATIC FORMS, INDEPENDENT OF THE AXIOM OF CHOICE

by S.G. DANI

A few years ago G.A. Margulis proved, thereby settling a longstanding conjecture of A. Oppenheim, that if Q is a nondegenerate indefinite quadratic form on $\mathbf{R}^n$, $n \geqslant 3$, which is not a scalar multiple of a rational form, then the set $Q(\mathbf{Z}^n)$ of values of Q on the set of integral n-tuples is a dense subset of $\mathbf{R}$ (cf. [M-1 and M-2]). In [DM-1] the result was strengthened, upholding density of the set of values of Q as above on the set of primitive integral n-tuples; an integral n-tuple is said to be primitive if there is no common divisor for the entries, other than ± 1. Subsequently, in [DM-2], an elementary proof was given for this result, depending only on standard arguments in topological groups and linear algebra. There are also some variations of the theme, in [M-3] and [S], giving a proof of a somewhat weaker result. All these proofs involve existence of minimal invariant subsets for various actions, which depends on Zorn's lemma and, in turn, on the axiom of choice (cf. [H], Theorem 7.10). Since the end result is arithmetical, it seems to be of interest to have a proof which does not depend on the axiom of choice. In this note we give a variation of the proof in [DM-2] which meets this objective; the modifications introduced for this purpose may also turn out to be useful in other ways.

As in [DM-2] the proof will be achieved by proving a result about orbits of certain actions on the homogeneous space $SL(3, \mathbf{R}) / SL(3, \mathbf{Z})$, namely the Theorem below. We shall formulate the result after introducing some notation; we choose it to be consistent with that in [DM-2] and refer to that paper whenever convenient, to avoid repetition. Let $G = SL(3, \mathbf{R})$ and $\Gamma = SL(3, \mathbf{Z})$. Let $\mathbf{R}^3$ be the 3-dimensional real vector space, viewed as the space of 3-rowed column vectors, equipped with the G-action by left multiplication. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbf{R}^3$. Let Q_0 be the quadratic form on $\mathbf{R}^3$ defined by $Q_0(p_1 e_1 + p_2 e_2 + p_3 e_3) = 2p_1 p_3 - p_2^2$ for all $p_1, p_2, p_3 \in \mathbf{R}$.

Let H be the subgroup of G consisting of all elements leaving Q_0 invariant; namely $H = \{g \in G \mid Q_0(gp) = Q_0(p)$ for all $p \in \mathbf{R}^3\}$. For each $t \in \mathbf{R}$ let $v_2(t)$ be the element of G such that $v_2(t)e_i = e_i$ *for* $i = 1$ and 2 and $v_2(t)e_3 = e_3 + te_1$. Let $V_2^+ = \{v_2(t) \mid t \geqslant 0\}$ and $V_2^- = \{v_2(t) \mid t \leqslant 0\}$. We prove the following:

THEOREM. *Let* $x \in G/\Gamma$ *and let* $X = \overline{Hx}$. *Then either* $X = Hx$ *or there exists a* $y \in G/\Gamma$ *such that* $V_2^+ y$ *or* $V_2^- y$ *is contained in* X.

As seen in [DM-2] the theorem implies Oppenheim's conjecture and in fact the following stronger result.

COROLLARY. *Let* Q *be a nondegenerate indefinite quadratic form on* $\mathbf{R}^n, n \geqslant 3$. *Suppose that* cQ *is not a rational quadratic form for any* $c > 0$. *Let* $\mathscr{P}(\mathbf{Z}^n)$ *denote the set of all primitive elements of* $\mathbf{Z}^n$. *Then* $Q(\mathscr{P}(\mathbf{Z}^n))$ *is a dense subset of* $\mathbf{R}$.

Before concluding this introduction it should be mentioned that the above theorem (and hence also the corollary) follows from Ratner's theorem proving Raghunathan's conjecture (cf. [R]; see also [DM-3] for a strengthening of the result and [M-4] for a general perspective of the area); however the proofs of Raghunathan's conjecture involve a considerably long argument (through several papers) and though it is believable that it could yield a proof of the above theorem independent of the axiom of choice, it is hard to ascertain this. On the other hand, the present proof only involves, apart from what is contained here, some results proved in [DM-2].

1. PRELIMINARIES

The remainder of the note deals with the proof of the Theorem. We begin with some more notation. Let B denote the subgroup consisting of all the upper triangular matrices in H. Let D^* be the subgroup consisting of all the diagonal matrices in B. Let D be the subgroup consisting of all the elements of D^* in which the diagonal entries are positive. Let V_1 be the subgroup consisting of all elements of B in which the diagonal entries are all 1; it is a one-parameter subgroup of H. Let $V_2 = \{v_2(t) \mid t \in \mathbf{R}\}$, where $v_2(t), t \in \mathbf{R}$, are as defined above. Let $V = V_1 V_2$; one can see that V is an abelian subgroup of G. It can also be verified that V_1 and V_2 are normalised by D^* and that $B = D^* V_1$. We note that D, V_1, V_2 and V defined here are the same as in [DM-2], where the subgroups are described by their matrix forms. Also, we denote by I the identity element in G. We now begin the proof with the following simple observation, analogues of which are involved in the earlier proofs as well, all the way from that in [M-1].

1.1. LEMMA. *Let Z be a closed H-invariant subset of G/Γ. Let $z \in Z$ be such that $\overline{V_1 z}$ is compact. Let $M = \{g \in G \mid gz \in Z\}$. Then for any $g \in \overline{HMV_1}$ there exists a $y \in \overline{V_1 z}$ such that $gy \in Z$.*

Proof. Indeed if $\{h_i\}$, $\{m_i\}$ and $\{u_i\}$ are sequences in H, M and V_1 respectively such that $h_i m_i u_i \to g$ and y is a limit point of $\{u_i^{-1} z\}$ then clearly $gy \in Z$.

Another major component is the following Proposition due to Margulis; (cf. Proposition 3 in [DM-2], where two different elementary proofs are given for the result).

1.2. PROPOSITION. *Let M be a subset of $G - HV_2$ such that $I \in \overline{M}$. Then either V_2^+ or V_2^- is contained in $\overline{HMV_1}$.*

We next recall from [DM-2] some more results that are needed here.

1.3. PROPOSITION (cf. [DM-2], Proposition A.12). *Any closed nonempty V_1-invariant subset of G/Γ contains a compact nonempty V_1-invariant subset.*

(We mention that, as in [DM-2], this result is not needed in the proof of the theorem in the case when $X = \overline{Hx}$ is known to be compact; the latter special case is adequate in proving the weaker version of the Corollary as dealt with in [M-3] and [S]; see [DM-2] for some details in this regard).

1.4. LEMMA. (i) *Any discrete subgroup of DV is either contained in V or generated by an element of vDv^{-1}, where $v \in V$.*

(ii) *If Δ is a discrete subgroup of $BV_2 = D^* V$ then either $\overline{V_1(V \cap \Delta)} = V$ or there exists a neighbourhood Θ of I in V_2 such that $B\Theta \cap \Delta \subseteq B$.*

Proof. Assertion (i) is the same as Lemma 5 of [DM-2]. Assertion (ii) can be deduced (i) by a simple computation; also, the relevant argument is available on page 157 of [DM-2], in the proof of case (c) of Proposition 8 there; we shall therefore not repeat it here.

1.5. PROPOSITION. (i) *HV_2 is a closed subset of G and the map $\eta: H \times V_2 \to HV_2$ defined by $\eta((h, v)) = hv$, for all $h \in H$ and $v \in V_2$, is a homeomorphism.*

(ii) *If $h \in H$ and there exists a $v \in V_2 - \{I\}$ such that $vh \in HV_2$ then $h \in B$.*

Proof. Observe that H is the isotropy subgroup of Q_0 under the contragradient action of G on the space of all quadratic forms on $\mathbf{R}^3$ (defined by $(g, Q) \mapsto Q^g$, where $Q^g(p) = Q(g^{-1}p)$ for all $g \in G$, quadratic forms Q and $p \in \mathbf{R}^3$). The V_2-orbit of Q_0 under the action can be explicitly written down (see [DM-2], page 148) and seen to be closed. Therefore V_2H is closed and hence so is HV_2, the latter being the same as $(V_2H)^{-1}$. Now consider the action of $H \times V_2$ on G where the element (h, v), with $h \in H$ and $v \in V_2$, acts by $g \mapsto hgv^{-1}$. Then HV_2 is the orbit of I and since it is closed and $H \cap V_2 = \{I\}$ it follows that $(h, v) \mapsto hv^{-1}$, $h \in H$ and $v \in V_2$, is a homeomorphism (cf. [MZ], Section 2.13, for instance). Hence so is η as in the hypothesis. This proves assertion (i). Assertion (ii) is precisely Proposition 4 from [DM-2].

For any $z \in G/\Gamma$ we denote by Γ_z the isotropy subgroup $\{g \in G \mid gz = z\}$ of z.

1.6. PROPOSITION. *Let $z \in G/\Gamma$ be a V_1-periodic point such that $H \cap \Gamma_z$ is not contained in B. Then Hz is a closed subset of G/Γ.*

Proof. Let Q be any quadratic form invariant under $H \cap \Gamma_z$. Since z is V_1-periodic there exists a $v \in V_1 - \{I\}$ such that $vz = z$. A straightforward computation using the v-invariance of Q then shows that Q is of the form $aQ_0 + bQ_1$ for some $a, b \in \mathbf{R}$, where Q_1 is the quadratic form on $\mathbf{R}^3$ defined by $Q_1(p_1e_1 + p_2e_2 + p_3e_3) = p_3^2$ for all $p_1, p_2, p_3 \in \mathbf{R}$. Now, Q and Q_0 are h-invariant for all $h \in H \cap \Gamma_z$, and hence so is bQ_1. If $b \neq 0$ this implies that $h \in B$ for all $h \in H \cap \Gamma_z$ (see the proof of Proposition 4 of [DM-2]). Since by hypothesis $H \cap \Gamma_z$ is not contained in B this implies that $b = 0$ and hence $Q = aQ_0$, with $a \in \mathbf{R}$.

Now let $g \in G$ be such that $z = g\Gamma$. Then in view of the above observation, any quadratic form invariant under $g^{-1}Hg \cap \Gamma = g^{-1}(H \cap \Gamma_z)g$ is a scalar multiple of the form Q' defined by $Q'(p) = Q_0(gp)$ for all $p \in \mathbf{R}^3$. The argument as on page 159 of [DM-2] (in the last part of the proof of Proposition 9 there) then implies that Q' is a scalar multiple of a rational form. Then the Γ-orbit of Q' under the contragradient action on the space of all quadratic forms on $\mathbf{R}^3$ is closed. This implies that $\Gamma g^{-1}Hg$ is closed and hence so is $g^{-1}Hg\Gamma$. This shows that $Hz = Hg\Gamma/\Gamma$ is closed, thus proving the proposition.

We recall that if φ is a homeomorphism of a topological space X then $x \in X$ is said to be a *recurrent* point for φ if there exists a sequence $\{n_k\}$ of natural numbers such that $n_k \to \infty$ and $n_k x \to x$. For the proof of the theorem we also need the following general fact.

1.7. PROPOSITION. *Let* φ *be a homeomorphism of a compact metric space* X. *Then there exists a recurrent point for* φ.

Proof. Given X and φ as in the hypothesis there exists a φ-invariant probability (Borel) measure on X (cf. [DGS], Proposition 3.8, for instance). The Proposition now follows from the Poincare recurrence theorem; see [M], Theorem 2.3, for a version of the Theorem in the form required here.

It can be seen, by perusing the proofs of the results quoted, from the references mentioned, that the above proof is indeed independent of the axiom of choice. For expositional purposes we also give in the Appendix a more self-contained proof of the Proposition. For this we use the same general idea as above but argue with invariant integrals (positive linear functionals on the space of continuous functions) constructed from the data, without actually using any measure theory.

Incidentally, it may be noted that the assertion in the Proposition is obvious if we assume Zorn's lemma, since in that case there exist compact minimal (nonempty) φ-invariant subsets of X and any point of such a subset is a recurrent point.

2. PROOF OF THE THEOREM

We will prove the Theorem after some technical preparation.

2.1. PROPOSITION. *Let* $x \in G/\Gamma$ *and* $X = \overline{Hx}$. *Let* $y \in X$ *and suppose that there exists a neighbourhood* Ω *of* I *in* G *such that* $\{g \in \Omega \mid gy \in X\} \subseteq HV_2$. *Then at least one of the following conditions holds:* (i) Hx *is open in* X *and* $y \in Hx$, (ii) Hy *and* DV_1y *are open in their (respective) closures or* (iii) $Vy \subseteq X$.

Proof. First suppose that in fact there exists a neighbourhood Ω' of I in G such that $\{g \in \Omega' \mid gy \in X\} \subseteq H$. Then Hy is open in X. Since Hx is dense in X it follows that $Hx = Hy$. Then Hx is open in X and $y \in Hx$, so condition (i) holds in this case. We may therefore assume that there does not exist any neighbourhood Ω' as above. In view of Proposition 1.5, (i), and the H-invariance of X this implies that there exists a sequence $\{v_i\}$ in $V_2 - \{I\}$ such that $v_i \to I$ and $v_i y \in X$ for all i.

Observe that if $V_1(V \cap \Gamma_y)$ is dense in V, then clearly $Vy \subseteq \overline{V_1 y} \subseteq X$ so condition (iii) is satisfied. We may therefore assume that it is not the case. Hence by Lemma 1.4, (ii), there exists a neighbourhood Θ of I in V_2 such that $B\Theta \cap \Gamma_y \subseteq B$. By replacing Ω as in the hypothesis by a smaller neighbourhood we may assume that Ω is open and $\Omega \cap HV_2 \subseteq (\Omega \cap H)\Theta$, the latter being possible because of Proposition 1.5, (i). Now let $g \in H$ be any element such that $gy \in \Omega y$; then there exist $h \in \Omega \cap H$ and $v \in \Theta$ such that $hv \in \Omega$ and $gy = hvy$. Hence $gv_i y = (gv_i g^{-1})gy = (gv_i g^{-1})hvy$. Since $gv_i g^{-1} \to I$ and Ω is a neighbourhood of hv it follows that $gv_i g^{-1}hv \in \Omega$ for all large i. Also $gv_i g^{-1}hvy = gv_i y \in X$ and hence by the hypothesis we get that for all large i, $gv_i g^{-1}hv \in HV_2$ and hence $v_i g^{-1}h \in HV_2$. Since $v_i \neq I$, for any i, by Proposition 1.5, (ii), this implies that $g^{-1}h \in B$. Then $g^{-1}hv \in B\Theta$. Also, since $gy = hvy, g^{-1}hv \in \Gamma_y$. By the choice of Θ these two conditions imply that $v = I$. Hence $gy = hy$. This shows that $Hy \cap \Omega y \subseteq (\Omega \cap H)y$. Similarly, since we had $g^{-1}h \in B$, it also shows that $By \cap \Omega y \subseteq (\Omega \cap B)y$. These conditions imply that Hy and By are open in their closures and since DV_1 is open in B it also follows that $DV_1 y$ is open in its closure; therefore condition (ii) holds in this case. This proves the Proposition.

2.2. PROPOSITION. *Let $x \in G/\Gamma$ be such that Hx is not closed and let $X = \overline{Hx}$. Let Y be a compact V_1-invariant subset of X and let $y \in Y$ be recurrent for the action of some $u \in V_1 - \{I\}$. Suppose that either Hx is not open in X or $y \notin Hx$. Then either $Vy \subseteq X$ or $I \in \overline{\{g \in G - HV_2 \mid gy \in X\}}$.*

Proof. Suppose that the assertion does not hold. Then Vy is not contained in X and there exists a neighbourhood Ω of I in G such that $\{g \in \Omega \mid gy \in X\} \subseteq HV_2$. Then Proposition 2.1 and the condition as in the hypothesis imply that Hy and $DV_1 y$ are open in their respective closures and $y \notin Hx$. Let $\Phi = DV_1 \cap \Gamma_y$. Since $DV_1 y$ is open in its closure, it is locally compact and hence $g\Phi \mapsto gy$ is a homeomorphism of DV_1/Φ on to $DV_1 y$ commuting with DV_1-action on the two spaces (cf. [MZ], Section 2.13, for instance). By hypothesis there exists a $u \in V_1 - \{I\}$ such that y is recurrent for the action of u. The preceding observation therefore implies that Φ is recurrent for the action of u on DV_1/Φ. It is easy to see that this can not happen if Φ is contained in vDv^{-1} for some $v \in V$. Applying Lemma 1.4, (i), we can conclude therefore that Φ is a nontrivial subgroup contained in V_1. Therefore y is a V_1-periodic point. Since $y \in X = \overline{Hx}$ and $\{g \in \Omega \mid gy \in X\} \subseteq HV_2$ it follows that there exists a sequence $\{v_i\}$ in V_2 such that $v_i \to I$ and $v_i y \in Hx$ for all i. For each i we have $v_i y \in Hx = Hv_1 y$ and therefore there exists a sequence $\{h_i\}$ in H such that $v_i y = h_i v_1 y$ for all i. Let $i \geqslant 1$ be arbitrary. Let $\Delta_i = H \cap \Gamma_{v_i y}$. Clearly Δ_i contains Φ and by the above relation it also contains $h_i \Phi h_i^{-1}$. Since $v_i y$ is V_1-periodic and $Hv_i y = Hx$ is not closed, by Proposition 1.6 Δ_i must be contained in B. Since $h_i \Phi h_i^{-1}$ is contained in Δ_i and consists of unipotent elements, this implies that $h_i \Phi h_i^{-1} \subseteq V_1$. This implies that $h_i \in B$ (since the subspaces spanned by $\{e_1\}$ and $\{e_1, e_2\}$ have to be h_i-invariant). Therefore there exist $d_i \in D^*$ and $u_i \in V_1$ such that $h_i = d_i u_i$. Now, since $h_i \Phi h_i^{-1} = \Gamma_{h_i v_1 y} \cap V_1 = \Gamma_{v_i y} \cap V_1 = \Phi$ and since Φ is a nontrivial subgroup of V_1 it follows that the diagonal entries of d_i are ± 1. Since $v_1 y$ is a V_1-periodic point, the preceding conclusion implies that the sequence $\{h_i v_1 y\}$ has a limit point in $Hv_1 y = Hx$. But $h_i v_1 y = v_i y \to y$ and therefore we get that $y \in Hx$, contradicting an earlier conclusion. This shows that the Proposition must hold.

Proof of the Theorem. We shall asssume that Hx is not closed and that X does not contain any V-orbit, since in either of these cases there is nothing more to be proved. Let $X' = X - Hx$ if Hx is open in X and $X' = X$ otherwise. Then X' is a closed nonempty V_1-invariant subset of X. By Propositions 2.2 and 1.7 any compact V_1-invariant subset of X' contains a y such that $I \in \overline{\{g \in G - HV_2 \mid gy \in X\}}$. Let $\{r_i\}$ be an enumeration of the set of all rational numbers. We now construct a decreasing sequence $\{Y_k\}$ of compact V_1-invariant subsets of X' and a sequence $\{t_k\}$ of rational numbers as follows. Recall that by Proposition 1.3 X' contains a compact nonempty V_1-invariant subset. Let Y_1 be such a subset and let $t_1 = 0$. After the sets $Y_1, \ldots, Y_k$ and the numbers $t_1, \ldots, t_k$ are chosen, for some $k \geqslant 1$, we proceed to choose Y_{k+1} and t_{k+1} as follows. As observed above, Y_k contains a point y such that $I \in \overline{M}$ where $M = \{g \in G - HV_2 \mid gy \in X\}$. Then by Proposition 1.2 $\overline{HMV_1}$ contains either V_2^+ or V_2^-. Now let i be the smallest natural number satisfying the following conditions: a) $r_i \neq t_j$ for any $j = 1, \ldots, k$ and b) r_i is positive if $V_2^+ \subseteq \overline{HMV_1}$ and negative otherwise. Put $t_{k+1} = r_i$. Then $v_2(t_{k+1}) \in \overline{HMV_1}$ and hence by Lemma 1.1 there exists a $y' \in \overline{V_1 y} \subseteq Y_k$ such that $v_2(t_{k+1})y' \subseteq X$. Put $Y_{k+1} = \overline{V_1 y'}$. This completes the inductive construction of the sequences $\{Y_k\}$ and $\{t_k\}$. It is clear from the construction that $\{Y_k\}$ is a decreasing sequence of compact

V_1-invariant subsets of X and $v_2(t_k)Y_k \subseteq X$ for all k. Also it is easy to see that $\{t_k \mid k \geqslant 1\}$ contains either all positive rational or all negative rational numbers. Now let $Y' = \cap_{k=1}^{\infty} Y_k$. Since $\{Y_k\}$ is a decreasing sequence of compact subsets, Y' is nonempty. Now if $\{t_k \mid k \geqslant 1\}$ contains all positive rational numbers then $v_2(r)Y' \subseteq X$ for all positive rational numbers r and hence by continuity $V_2^+ Y' \subseteq X$ and, similarly, in the alternative case $V_2^- Y' \subseteq X$. This completes the proof of the theorem.

APPENDIX: RECURRENT POINTS

For a compact metric space X we denote by $C(X)$ the space of all continuous real-valued functions on X equipped with the sup-norm topology and by $C(X)^+$ the subset of $C(X)$ consisting of all nonnegative functions; the supremum norm of $f \in C(X)$, namely $\sup\{|f(x)| \mid x \in X\}$, will be denoted by $\|f\|$. By an integral on $C(X)$ we mean a linear functional on $C(X)$ which takes nonnegative values on $C(X)^+$. For an integral Λ on $C(X)$ the *support* of Λ is defined to be the subset of X consisting of all $x \in X$ such that $\Lambda(f) > 0$ for any $f \in C(X)^+$ for which $f(x) > 0$; the support is easily seen to be a closed subset of X. It can also be verified by a simple point-set topological argument that if Λ is an integral on $C(X)$ and $f \in C(X)$ vanishes on the support of Λ then $\Lambda(f) = 0$. If Λ is an integral on $C(X)$, where X is a compact metrizable space, and X' is the support of Λ then there exists a unique integral Λ' on $C(X')$ such that $\Lambda'(f|_{X'}) = \Lambda(f)$ for all $f \in C(X)$, where $f|_{X'}$ denotes the restriction of f to X'; this follows from the Tietze-Urysohn extension theorem (cf. [D], (4.5.1)) and the above mentioned property of the support. We note also that the support of Λ' as above is the whole of X'.

For any homeomorphism φ of a compact (metrizable) space X an integral Λ on $C(X)$ is said to be φ-invariant if $\Lambda(f \circ \varphi) = \Lambda(f)$ for all $f \in C(X)$; clearly the support of a φ-invariant integral on $C(X)$ is a φ-invariant (closed) subset of X.

Proof of Proposition 1.7. We fix a dense sequence in $C(X)$, say f_j, $j = 1, 2, \ldots$. Let $x_0 \in X$. Given f_j, for any sequence $\{m_k\}$ of natural numbers $m_k^{-1} \sum_{i=0}^{m_k-1} f_j \circ \varphi^i(x_0)$ is a bounded sequence and therefore admits a convergent subsequence. Using a standard procedure (finding $\{m_k^{(j)}\}$, with each sequence a subsequence of the previous one, such that the corresponding sequence for f_j as above converges and considering $\{m_k^{(k)}\}$) we get a sequence $\{n_k\}$ of natural numbers such that $n_k^{-1} \sum_{i=0}^{n_k-1} f_j \circ \varphi^i(x_0)$ converges for all j; also, the limit is between $-\|f_j\|$ and $\|f_j\|$. Since $\{f_j\}$ is dense in $C(X)$ this readily implies that $n_k^{-1} \sum_{i=0}^{n_k - 1} f \circ \varphi^i(x_0)$ converges for all $f \in C(X)$; let c_f be the limit corresponding to f. Then it can be verified that $\Lambda: C(X) \to \mathbf{R}$ defined by $\Lambda(f) = c_f$, for all $f \in C(X)$, is a φ-invariant integral on $C(X)$. Also clearly Λ is not identically zero and therefore by our observations above, the support, say X', is a nonempty closed φ-invariant subset of X and further $C(X')$ admits an integral with full support (namely X') which is invariant under the restriction of φ to X'. Replacing X as in the hypothesis by X' we may without loss of generality assume that $C(X)$ admits a φ-invariant integral whose support is X; in the rest of the argument we let Λ be any such integral.

Now suppose that there do not exist any recurrent points for φ. Let $\rho(\cdot, \cdot)$ be the metric on X. Let θ be the function on X defined by $\theta(x) = \inf\{\rho(\varphi^i(x), x) \mid i = 1, 2, \ldots\}$, for all $x \in X$. There being no recurrent points means that $\theta(x) > 0$ for all $x \in X$. For each natural number k let $E_k = \{x \in X \mid \theta(x) \geqslant 1/k\}$. Then each E_k is a closed subset of X and $X = \cup E_k$. Therefore by the Baire category theorem there exists a k such that E_k has an interior point in X. In particular, there exists an open ball, say A, of radius at most $1/3k$ contained in E_k. The definition of E_k and the condition on the radius of A then imply that the sets $\varphi^i(A)$, $i \in \mathbf{Z}$, are mutually disjoint. Now let $x \in A$ and let $f \in C(X)^+$ be such that $f(x) > 0$ and the support of f (the closure of the set $\{y \in X \mid f(y) > 0\}$) is contained in A. For each natural number n let $S_n(f) = \sum_{i=0}^{n-1} f \circ \varphi^i \in C(X)$. The disjointness of $\varphi^i(A)$, $i \in \mathbf{Z}$, implies that, for any n, $\| S_n(f) \| = \| f \|$. Also, by the φ-invariance of Λ we have $\Lambda(S_n(f)) = n\Lambda(f)$. Hence $\Lambda(f) = \Lambda(S_n(f))/n \leqslant \| S_n(f) \| \Lambda(1_X)/n = \| f \| \Lambda(1_X)/n$ for all n, where 1_X denotes the constant function with value 1. But this implies that $\Lambda(f) = 0$ contradicting the assumption that the support of Λ is the whole of X. This proves the proposition.

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REFERENCES

[DM-1] Dani, S.G. and G.A. Margulis. Values of quadratic forms at primitive integral points. *Invent. Math. 98* (1989), 405-424.

[DM-2] Dani, S.G. and G.A. Margulis. Values of quadratic forms at integral points; an elementary approach. *L'Enseignement Math. 36* (1990), 143-174.

[DM-3] DANI, S.G. and G.A. MARGULIS. Limit distributions of orbits of unipotent flows and values of quadratic forms. *Advances in Soviet Math. 16* (1993), 91-137.

[DGS] DENKER, M., C. GRILLENBERGER and K. SIGMUND. *Ergodic Theory on Compact Spaces*. Springer-Verlag, 1976.

[D] DIEUDONNÉ, J. *Foundations of Modern Analysis*. Academic Press, 1969.

[H] HENLE, J.M. *An Outline of Set Theory*. Springer-Verlag, 1986.

[M] MAÑÉ, R. *Ergodic Theory and Differentiable Dynamics*. Springer-Verlag, 1983.

[M-1] MARGULIS, G.A. Formes quadratiques indéfinies et flots unipotents sur les espaces homogènes. *C.R. Acad. Sci. Paris, Série I, 304* (1987), 249-253.

[M-2] —— Discrete subgroups and ergodic theory; in: *Number Theory, Trace Formulas and Discrete Subgroups*. Academic Press, 1989.

[M-3] —— Orbits of group actions and values of quadratic forms at integral points. *Israel J. Math. 3* (1990), 127-150.

[M-4] —— Dynamical and ergodic properties of subgroup actions on homogeneous spaces with applications to number theory; in *Proceedings of the International Congress of Mathematicians (Kyoto 1990),* The Mathematical Society of Japan, Tokyo; Springer-Verlag, 1991.

[MZ] MONTGOMERY, D. and L. ZIPPIN. *Topological Transformations Groups*. Interscience Publishers, New York, 1955.

[R] RATNER, M. Raghunathan's topological conjecture and distributions of unipotent flows, *Duke Math. J. 63* (1991), 235-280.

[S] SIKORAV, J.-C. Valeurs des formes quadratiques indéfinies irrationelles; in *Séminaire de Théorie des Nombres, Paris 1987-88,* Progress in Math. 81, Birkhauser 1990.

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