1. Preliminaries

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Let H be the subgroup of G consisting of all elements leaving Q_0 invariant; namely $H = \{g \in G \mid Q_0(gp) = Q_0(p)$ for all $p \in \mathbf{R}^3\}$. For each $t \in \mathbf{R}$ let $v_2(t)$ be the element of G such that $v_2(t)e_i = e_i$ *for* $i = 1$ and 2 and $v_2(t)e_3 = e_3 + te_1$. Let $V_2^+ = \{v_2(t) \mid t \geqslant 0\}$ and $V_2^- = \{v_2(t) \mid t \leqslant 0\}$. We prove the following:

THEOREM. *Let* $x \in G/\Gamma$ *and let* $X = \overline{Hx}$. *Then either* $X = Hx$ *or there exists a* $y \in G/\Gamma$ *such that* $V_2^+ y$ *or* $V_2^- y$ *is contained in* X.

As seen in [DM-2] the theorem implies Oppenheim's conjecture and in fact the following stronger result.

COROLLARY. *Let* Q *be a nondegenerate indefinite quadratic form on* $\mathbf{R}^n$, $n \geqslant 3$. *Suppose that* cQ *is not a rational quadratic form for any* $c > 0$. *Let* $\mathscr{P}(\mathbf{Z}^n)$ *denote the set of all primitive elements of* $\mathbf{Z}^n$. *Then* $Q(\mathscr{P}(\mathbf{Z}^n))$ *is a dense subset of* $\mathbf{R}$.

Before concluding this introduction it should be mentioned that the above theorem (and hence also the corollary) follows from Ratner's theorem proving Raghunathan's conjecture (cf. [R]; see also [DM-3] for a strengthening of the result and [M-4] for a general perspective of the area); however the proofs of Raghunathan's conjecture involve a considerably long argument (through several papers) and though it is believable that it could yield a proof of the above theorem independent of the axiom of choice, it is hard to ascertain this. On the other hand, the present proof only involves, apart from what is contained here, some results proved in [DM-2].

1. PRELIMINARIES

The remainder of the note deals with the proof of the Theorem. We begin with some more notation. Let B denote the subgroup consisting of all the upper triangular matrices in H. Let D^* be the subgroup consisting of all the diagonal matrices in B. Let D be the subgroup consisting of all the elements of D^* in which the diagonal entries are positive. Let V_1 be the subgroup consisting of all elements of B in which the diagonal entries are all 1; it is a one-parameter subgroup of H. Let $V_2 = \{v_2(t) \mid t \in \mathbf{R}\}$, where $v_2(t), t \in \mathbf{R}$, are as defined above. Let $V = V_1 V_2$; one can see that V is an abelian subgroup of G. It can also be verified that V_1 and V_2 are normalised by D^* and that $B = D^* V_1$. We note that D, V_1, V_2 and V defined here are the same as in [DM-2], where the subgroups are described by their matrix forms. Also, we denote by I the identity element in G. We now begin the proof with the following simple observation, analogues of which are involved in the earlier proofs as well, all the way from that in [M-1].

1.1. LEMMA. *Let* Z *be a closed* H*-invariant subset of* G/Γ. *Let* $z \in Z$ *be such that* $\overline{V_1 z}$ *is compact. Let* $M = \{g \in G \mid gz \in Z\}$. *Then for any* $g \in \overline{HMV_1}$ *there exists a* $y \in \overline{V_1 z}$ *such that* $gy \in Z$.

Proof. Indeed if $\{h_i\}$, $\{m_i\}$ and $\{u_i\}$ are sequences in H, M and V_1 respectively such that $h_i m_i u_i \to g$ and y is a limit point of $\{u_i^{-1} z\}$ then clearly $gy \in Z$.

Another major component is the following Proposition due to Margulis; (cf. Proposition 3 in [DM-2], where two different elementary proofs are given for the result).

1.2. PROPOSITION. *Let* M *be a subset of* $G - HV_2$ *such that* $I \in \overline{M}$. *Then either* V_2^+ *or* V_2^- *is contained in* $\overline{HMV_1}$.

We next recall from [DM-2] some more results that are needed here.

1.3. PROPOSITION (cf. [DM-2], Proposition A.12). *Any closed nonempty* V_1*-invariant subset of* G/Γ *contains a compact nonempty* V_1*-invariant subset.*

(We mention that, as in [DM-2], this result is not needed in the proof of the theorem in the case when $X = \overline{Hx}$ is known to be compact; the latter special case is adequate in proving the weaker version of the Corollary as dealt with in [M-3] and [S]; see [DM-2] for some details in this regard).

1.4. LEMMA. (i) *Any discrete subgroup of* DV *is either contained in* V *or generated by an element of* vDv^{-1}, *where* $v \in V$.

(ii) *If* Δ *is a discrete subgroup of* $BV_2 = D^*V$ *then either* $\overline{V_1(V \cap \Delta)} = V$ *or there exists a neighbourhood* Θ *of* I *in* V_2 *such that* $B\Theta \cap \Delta \subseteq B$.

Proof. Assertion (i) is the same as Lemma 5 of [DM-2]. Assertion (ii) can be deduced (i) by a simple computation; also, the relevant argument is available on page 157 of [DM-2], in the proof of case (c) of Proposition 8 there; we shall therefore not repeat it here.

1.5. PROPOSITION. (i) *HV_2 is a closed subset of G and the map $\eta: H \times V_2 \to HV_2$ defined by $\eta((h, v)) = hv$, for all $h \in H$ and $v \in V_2$, is a homeomorphism.*

(ii) *If $h \in H$ and there exists a $v \in V_2 - \{I\}$ such that $vh \in HV_2$ then $h \in B$.*

Proof. Observe that H is the isotropy subgroup of Q_0 under the contragradient action of G on the space of all quadratic forms on $\mathbf{R}^3$ (defined by $(g, Q) \mapsto Q^g$, where $Q^g(p) = Q(g^{-1}p)$ for all $g \in G$, quadratic forms Q and $p \in \mathbf{R}^3$). The V_2-orbit of Q_0 under the action can be explicitly written down (see [DM-2], page 148) and seen to be closed. Therefore V_2H is closed and hence so is HV_2, the latter being the same as $(V_2H)^{-1}$. Now consider the action of $H \times V_2$ on G where the element (h, v), with $h \in H$ and $v \in V_2$, acts by $g \mapsto hgv^{-1}$. Then HV_2 is the orbit of I and since it is closed and $H \cap V_2 = \{I\}$ it follows that $(h, v) \mapsto hv^{-1}$, $h \in H$ and $v \in V_2$, is a homeomorphism (cf. [MZ], Section 2.13, for instance). Hence so is η as in the hypothesis. This proves assertion (i). Assertion (ii) is precisely Proposition 4 from [DM-2].

For any $z \in G/\Gamma$ we denote by Γ_z the isotropy subgroup $\{g \in G \mid gz = z\}$ of z.

1.6. PROPOSITION. *Let $z \in G/\Gamma$ be a V_1-periodic point such that $H \cap \Gamma_z$ is not contained in B. Then Hz is a closed subset of G/Γ.*

Proof. Let Q be any quadratic form invariant under $H \cap \Gamma_z$. Since z is V_1-periodic there exists a $v \in V_1 - \{I\}$ such that $vz = z$. A straightforward computation using the v-invariance of Q then shows that Q is of the form $aQ_0 + bQ_1$ for some $a, b \in \mathbf{R}$, where Q_1 is the quadratic form on $\mathbf{R}^3$ defined by $Q_1(p_1e_1 + p_2e_2 + p_3e_3) = p_3^2$ for all $p_1, p_2, p_3 \in \mathbf{R}$. Now, Q and Q_0 are h-invariant for all $h \in H \cap \Gamma_z$, and hence so is bQ_1. If $b \neq 0$ this implies that $h \in B$ for all $h \in H \cap \Gamma_z$ (see the proof of Proposition 4 of [DM-2]). Since by hypothesis $H \cap \Gamma_z$ is not contained in B this implies that $b = 0$ and hence $Q = aQ_0$, with $a \in \mathbf{R}$.

Now let $g \in G$ be such that $z = g\Gamma$. Then in view of the above observation, any quadratic form invariant under $g^{-1}Hg \cap \Gamma = g^{-1}(H \cap \Gamma_z)g$ is a scalar multiple of the form Q' defined by $Q'(p) = Q_0(gp)$ for all $p \in \mathbf{R}^3$. The argument as on page 159 of [DM-2] (in the last part of the proof of Proposition 9 there) then implies that Q' is a scalar multiple of a rational form. Then the Γ-orbit of Q' under the contragradient action on the space of all quadratic forms on $\mathbf{R}^3$ is closed. This implies that $\Gamma g^{-1}Hg$ is closed and hence so is $g^{-1}Hg\Gamma$. This shows that $Hz = Hg\Gamma/\Gamma$ is closed, thus proving the proposition.

We recall that if φ is a homeomorphism of a topological space X then $x \in X$ is said to be a *recurrent* point for φ if there exists a sequence $\{n_k\}$ of natural numbers such that $n_k \to \infty$ and $n_k x \to x$. For the proof of the theorem we also need the following general fact.

1.7. PROPOSITION. *Let φ be a homeomorphism of a compact metric space X. Then there exists a recurrent point for φ.*

Proof. Given X and φ as in the hypothesis there exists a φ-invariant probability (Borel) measure on X (cf. [DGS], Proposition 3.8, for instance). The Proposition now follows from the Poincare recurrence theorem; see [M], Theorem 2.3, for a version of the Theorem in the form required here.

It can be seen, by perusing the proofs of the results quoted, from the references mentioned, that the above proof is indeed independent of the axiom of choice. For expositional purposes we also give in the Appendix a more self-contained proof of the Proposition. For this we use the same general idea as above but argue with invariant integrals (positive linear functionals on the space of continuous functions) constructed from the data, without actually using any measure theory.

Incidentally, it may be noted that the assertion in the Proposition is obvious if we assume Zorn's lemma, since in that case there exist compact minimal (nonempty) φ-invariant subsets of X and any point of such a subset is a recurrent point.

2. PROOF OF THE THEOREM

We will prove the Theorem after some technical preparation.

2.1. PROPOSITION. *Let $x \in G/\Gamma$ and $X = \overline{Hx}$. Let $y \in X$ and suppose that there exists a neighbourhood Ω of I in G such that $\{g \in \Omega \mid gy \in X\} \subseteq HV_2$. Then at least one of the following conditions holds:* (i) *Hx is open in X and $y \in Hx$,* (ii) *Hy and $DV_1 y$ are open in their (respective) closures or* (iii) *$Vy \subseteq X$.*

Proof. First suppose that in fact there exists a neighbourhood Ω' of I in G such that $\{g \in \Omega' \mid gy \in X\} \subseteq H$. Then Hy is open in X. Since Hx is dense in X it follows that $Hx = Hy$. Then Hx is open in X and $y \in Hx$, so