

# 4. Weight enumerators of finite scalar product modules

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

The associated Witt class is

$$w(\mathbf{E}_6) = \langle 1 \rangle \quad \text{in} \quad W(\mathbf{F}_3) .$$

CASE  $R = \mathbf{E}_7$ .

The definition is

$$\mathbf{ZE}_7 = \{ \sum_{i=1}^8 x_i e_i : 2x_i \in \mathbf{Z}, x_i - x_j \in \mathbf{Z}, \sum_{i=1}^8 x_i = 0 \} .$$

Here,

$$(\mathbf{ZE}_7)^\# = \mathbf{ZE}_7 \sqcup (\mathbf{ZE}_7 + z_1) ,$$

where

$$z_1 = \frac{1}{4} (e_1 + e_2 + e_3 + e_4 + e_5 + e_6 - 3(e_7 + e_8))$$

satisfies  $(z_1, z_1) = \frac{3}{2}$  and is of minimal scalar square in its class *mod*  $\mathbf{ZE}_7$ .

Again,  $z_1$  is noted  $x_1(\mathbf{E}_7)$  if convenient.

The Witt class  $w(\mathbf{E}_7)$  is the generator  $\langle 1 \rangle$  of  $W(\mathbf{F}_2) = \mathbf{Z}/2\mathbf{Z}$ .

CASE  $R = \mathbf{E}_8$ .

Here,  $T(\mathbf{E}_8) = 0$ . The associated Witt class is 0.

#### 4. WEIGHT ENUMERATORS OF FINITE SCALAR PRODUCT MODULES

Let  $T$  be a finite abelian group with a non-degenerate bilinear form  $b : T \times T \rightarrow \mathbf{Q}/\mathbf{Z}$ .

Suppose that we have a decomposition of  $T$  as an orthogonal direct sum of subgroups  $T_1, \dots, T_s$ :

$$T = T_1 \boxplus T_2 \boxplus \dots \boxplus T_s .$$

Then we can define the weight  $x^{w(u)} \in \mathbf{Z}[x_1, \dots, x_s]$  of an element  $u \in T$  by tabulating its non-zero components in the decomposition  $u = u_1 + u_2 + \dots + u_s$ ,  $u_i \in T_i$ , as

$$x^{w(u)} = x_1^{w(u_1)} \cdot x_2^{w(u_2)} \cdot \dots \cdot x_s^{w(u_s)} ,$$

where

$$w(u_i) = \begin{cases} 0 & \text{if } u_i = 0, \\ 1 & \text{if } u_i \neq 0. \end{cases}$$

If  $M$  is a subset of  $T$ , the *weight enumerator* of  $M$  is the polynomial

$$P_M(x_1, \dots, x_s) = \sum_{u \in M} x^{w(u)}.$$

We denote by  $q_i$ ,  $i = 1, \dots, s$  the order of the subgroup  $T_i$ .

We show in this section that MacWilliams duality is still valid in this more general setting:

**THEOREM.** *Let  $M \subset T$  be a subgroup of the scalar product module  $T = T_1 \oplus T_2 \oplus \dots \oplus T_s$ . Set  $q_i = \text{Card}(T_i)$ , and let  $M^\perp$  be the subgroup orthogonal to  $M$ . Then, we have the formula, where  $|M| = \text{Card}(M)$ :*

$$P_{M^\perp}(x_1, \dots, x_s) = \frac{1}{|M|} \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot P_M\left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_s}{1 + (q_s - 1)x_s}\right).$$

Note that if some of the subgroups  $T_1, \dots, T_s$  are mutually isomorphic (or more generally have the same order), then we can write the decomposition of  $T$  in the form

$$T = n_1 T_1 \oplus n_2 T_2 \oplus \dots \oplus n_r T_r,$$

where  $n_i T_i$  stands for the orthogonal sum

$$n_i T_i = T_i \oplus T_i \oplus \dots \oplus T_i$$

of  $n_i$  copies of  $T_i$ .

The weight of an element

$$u = (u_{1,1} + \dots + u_{1,n_1}) + \dots + (u_{r,1} + \dots + u_{r,n_r})$$

is then defined as

$$x^{w(u)} = x_1^{v_1} \cdot x_2^{v_2} \cdot \dots \cdot x_r^{v_r},$$

where  $v_i$  is the number of non-zero components of  $u_{i,1} + \dots + u_{i,n_i}$  in  $n_i T_i$ .

The duality theorem then takes the seemingly more general form

$$P_{M^\perp}(x_1, \dots, x_r) = \frac{1}{\text{Card}(M)} \prod_{i=1}^r (1 + (q_i - 1)x_i)^{n_i} \cdot P_M\left(\frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_r}{1 + (q_r - 1)x_r}\right).$$

This identity can be viewed as a system of linear equations for the coefficients of the weight enumerator polynomial  $P_M$  of any putative metabolizer  $M = M^\perp$ . If  $M$  exists, this system must be solvable in non-negative integers.

*Proof of the duality theorem.* One of the classical proofs of MacWilliams duality in a vector space over a finite field goes over with only insignificant changes. We repeat the argument for the reader's convenience.

Let  $\chi: \mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{C}^*$  be the character given by  $\chi(\alpha) = e^{2\pi i\alpha}$ . Set  $\beta(u, v) = \chi(b(u, v))$ .

We cook up the function  $f: T \rightarrow \mathbf{C}[x_1, \dots, x_s]$  given by

$$f(u) = \sum_{v \in T} \beta(u, v) \cdot x^{w(v)}$$

and evaluate  $\sum_{u \in M} f(u)$  in two different ways, using the following lemma:

LEMMA.

$$\sum_{u \in M} \beta(u, v) = \begin{cases} \text{Card}(M) & \text{if } v \in M^\perp, \\ 0 & \text{if } v \notin M^\perp. \end{cases}$$

We first recall the proof of the lemma.

If  $v \in M^\perp$ , then  $\beta(u, v) = 1$  for every  $u \in M$ , thus  $\sum_{u \in M} \beta(u, v) = \text{Card}(M)$  as stated in this case.

If  $v \notin M^\perp$ , there is an element  $u_1 \in M$  such that  $b(u_1, v) \neq 0$ , and then  $\beta(u_1, v) \neq 1$ . We have

$$\begin{aligned} \sum_{u \in M} \beta(u, v) &= \sum_{u \in M} \beta(u_1 + u, v) \\ &= \sum_{u \in M} \beta(u_1, v) \beta(u, v) = \beta(u_1, v) \sum_{u \in M} \beta(u, v). \end{aligned}$$

This implies the statement of the lemma for  $v \notin M^\perp$ .

We now proceed to the proof of the duality theorem.

Firstly,

$$\begin{aligned} \sum_{u \in M} f(u) &= \sum_{u \in M} \sum_{v \in T} \beta(u, v) \cdot x^{w(v)} = \sum_{v \in T} \left( \sum_{u \in M} \beta(u, v) \right) \cdot x^{w(v)} \\ &= \sum_{v \in M^\perp} \text{Card}(M) \cdot x^{w(v)} = \text{Card}(M) \cdot P_{M^\perp}(x_1, \dots, x_s). \end{aligned}$$

Secondly,

$$\begin{aligned} f(u) &= \sum_{v \in T} \beta(u, v) \cdot x^{w(v)} \\ &= \sum_{v_1 \in T_1, \dots, v_s \in T_s} \beta(u_1, v_1) \cdot \dots \cdot \beta(u_s, v_s) \cdot x_1^{w(v_1)} \cdot \dots \cdot x_s^{w(v_s)} \\ &= \prod_{i=1}^s \left( \sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)} \right), \end{aligned}$$

where  $u = u_1 + \dots + u_s$  is the decomposition of  $u \in T = T_1 \boxplus \dots \boxplus T_s$ .

Using the lemma again, we have

$$\sum_{v \in T_i} \beta(u_i, v) \cdot x_i^{w(v)} = \begin{cases} 1 + (q_i - 1)x_i & \text{if } u_i = 0, \\ 1 - x_i & \text{if } u_i \neq 0. \end{cases}$$

Thus,

$$f(u) = \prod_{i \in S} (1 + (q_i - 1)x_i) \cdot \prod_{i \in S'} (1 - x_i),$$

where  $S \subset \{1, \dots, s\}$  is the set of indices  $i$  for which  $u_i = 0$ , and  $S' \subset \{1, \dots, s\}$  the set of indices  $i$  for which  $u_i \neq 0$ .

Another way of writing  $f(u)$  is

$$f(u) = \prod_{i=1}^s (1 - x_i)^{w(u_i)} \cdot (1 + (q_i - 1)x_i)^{1 - w(u_i)}.$$

Plugging this formula into  $\sum_{u \in M} f(u)$ , we get

$$\begin{aligned} \sum_{u \in M} f(u) &= \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot \sum_{u \in M} \prod_{i=1}^s \left( \frac{1 - x_i}{1 + (q_i - 1)x_i} \right)^{w(u_i)} \\ &= \prod_{i=1}^s (1 + (q_i - 1)x_i) \cdot P_M \left( \frac{1 - x_1}{1 + (q_1 - 1)x_1}, \dots, \frac{1 - x_s}{1 + (q_s - 1)x_s} \right). \end{aligned}$$

Comparing the two expressions for  $\sum_{u \in M} f(u)$ , we get the theorem.

## 5. THE DEFICIENCY

The main further necessary condition for a root system to be contained in an even unimodular lattice of the same rank is provided by the notion of deficiency (Defekt) introduced and studied in [KV].

If  $R$  is a root system of rank  $n$ , the *deficiency* of  $R$ , denoted  $d(R)$ , is the difference  $n - m$ , where  $m$  is the maximal cardinality of a set  $\{a_1, \dots, a_m\} \subset R$  of mutually orthogonal roots

$$(a_i, a_j) = 2\delta_{ij}, \quad \text{for all } 1 \leq i, j \leq m.$$

We use this notion only if all roots in  $R$  have the same scalar square 2.

If  $R = R_1 \boxplus R_2$ , then  $d(R) = d(R_1) + d(R_2)$ . The values of the deficiency for the irreducible root systems are

$$\begin{aligned} d(\mathbf{A}_l) &= \left[ \frac{l}{2} \right], \\ d(\mathbf{D}_l) &= \begin{cases} 0 & \text{for } l \text{ even,} \\ 1 & \text{for } l \text{ odd,} \end{cases} \end{aligned}$$

$$d(\mathbf{E}_6) = 2, \quad d(\mathbf{E}_7) = d(\mathbf{E}_8) = 0.$$