6. The tables

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By Satz 5 of [KV], if R is the (complete) root system of an even unimodular lattice of rank 32, then

$$d(R) = 0, 8, 12, 14, 15 \text{ or } 16 .$$

The proof consists in constructing from the given lattice a new lattice L, still of rank 32 and containing the orthogonal sum of $m = 32 - d(R)$ copies of $\mathbf{Z}$. Thus, $L = \mathbf{Z}^m \boxplus L_0$, where L_0 is again unimodular and of rank $d(R)$. (Hence, $\mathrm{rank}(L_0) \leqslant 16$.)

By Martin Kneser's classification of unimodular (positive definite) lattices of rank $\leqslant 16$, the rank of L_0, i.e. $d(R)$ can only take the above values. (See [Kn], Satz 1.)

In setting up the tables we conveniently use the deficiency to discriminate the various root systems R according to the value of $d(R)$.

6. THE TABLES

We now proceed to list the *indecomposable* even unimodular lattices L of rank 32 with a complete root system R.

The presence in R of a factor of type $\mathbf{E}_8$ would produce a unimodular sublattice $\mathbf{Z}\mathbf{E}_8 = L_0 \subset L$, and hence a decomposition $L = L_0 \boxplus L_1$ for some (even) unimodular L_1 of rank 24. Hence, we assume throughout that R has the form

$$R = \mathbf{A}_{l_1} \boxplus \ldots \boxplus \mathbf{A}_{l_r} \boxplus \mathbf{D}_{m_1} \boxplus \ldots \boxplus \mathbf{D}_{m_s} \boxplus \mathbf{mE}_6 \boxplus \mathbf{nE}_7 ,$$

with no factor of type $\mathbf{E}_8$.

Altogether there are $N = 88523$ such systems (of rank 32). The possible dimensions for $\mathbf{mE}_6 \boxplus \mathbf{nE}_7$ are

$$D = \{0, 6, 7, 12, 13, 14, 18, 19, 20, 21, 24, 25, 26, 27, 28, 30, 31, 32\}$$

and for $d \in D$, there is a unique pair (m, n) such that $d = 6m + 7n$. Hence

$$N = \textstyle\sum_{d \in D} \sum_{i=0}^{32-d} p(i) q(32 - d - i) ,$$

where $p(i)$ is the number of partitions of i and $q(j)$ is the number of partitions $(j_1, \ldots, j_t)$ of j with $4 \leqslant j_1 \leqslant \ldots \leqslant j_t$. (Of course, we use the convention $p(0) = q(0) = 1$.)

Among these, only 21209 have an acceptable deficiency, i.e. $d = 0, 8, 12, 14, 15$ or 16. They are distributed as follows:

Deficiency	0	8	12	14	15	16	**Total**
Number	347	9799	6282	3027	1523	231	21209
Number with zero Witt class	347	848	306	90	57	28	1676
Number of connected root systems with zero Witt class	347	410	108	34	24	11	934

We say that a root system R is *not connected* if $R = R_1 \sqcup R_2$ is a disjoint union of mutually orthogonal root systems R_1, R_2 such that $T(R_1)$ and $T(R_2)$ have relatively prime orders.

If $R = R_1 \sqcup R_2$ is not connected, a metabolizer for $T(R) = T(R_1) \boxplus T(R_2)$ will have the form $M = M_1 \boxplus M_2$, where M_i is a metabolizer for $T(R_i)$, $i = 1, 2$ and any lattice L with (complete) root system R will split as $L = L_1 \boxplus L_2$, with L_1, L_2 unimodular and with root systems R_1, R_2 respectively. Thus, if R is not connected, it does not qualify as a candidate root system for an indecomposable unimodular lattice of the same rank.

Sifting the root systems for the purpose of setting up the tables, we retain only the connected ones. Of course, a decomposable 32-dimensional lattice which does not involve a $\mathbf{ZE}_8$ factor can only be the orthogonal sum of 2 copies of the indecomposable 16-dimensional lattice Γ_{16} in the notation of [MH], Lemma 6.1, p. 27. However, the criterion is a handy one to include in a computer program and it does considerably shorten the lists of candidates. The number of remaining systems is shown as the last line in the above table.

In order to get some experimental estimate on the relative strengths of the various conditions we are using, let me display the (otherwise irrelevant) list of connected systems of admissible deficiencies. *(See the table next page.)*

Comparing the last lines of the two tables we see that the condition on the Witt class is fairly stronger than merely requiring the order of $T(R)$ to be an integral square. (Of course, if $T(R)$ contains a metabolizer $M = M^\perp$, then $|T(R)| = |M|^2$.) A simple example of a root system R with non-zero Witt class but $|T(R)|$ a square is $R = 2\mathbf{A}_5 \boxplus \mathbf{A}_8 \boxplus \mathbf{D}_4 \boxplus \mathbf{D}_8$ which is connected (and has deficiency 8). There are $1302 - 934 = 368$ such.

Deficiency	0	8	12	14	15	16	Total
Connected root systems	347	2154	1051	425	150	25	4152
Connected root systems with $\mid T(R)\mid$ *a square*	347	610	214	79	38	14	1302

The 934 root systems of the bottom row of the first table all possess a metabolizer. However, a metabolizer $M \subset T(R)$ will produce a unimodular lattice L with root system exactly R only if for each non-zero $s \in M$ the norm $\mathbf{n}(s)$ is an integer larger than 2: $\mathbf{n}(s) > 2$. (The norm has been defined in Section 2.) Moreover if L is to be an even lattice, $\mathbf{n}(s)$ must in addition be an even integer. A metabolizer M satisfying $\mathbf{n}(s) \equiv 0 \ (mod\, 2)$ and $\mathbf{n}(s) > 2$ for every $s \in M$, $s \neq 0$ will be called *admissible.*

The norms of the elements of $T(\mathbf{A}_l)$, $T(\mathbf{D}_l)$, $T(\mathbf{E}_6)$, and $T(\mathbf{E}_7)$ have been recalled in Section 3:

$$\mathbf{n}(x_r) = \tfrac{r(l+1-r)}{l+1} \text{ for } x_r \in T(\mathbf{A}_l),\ r = 0, 1, \ldots, l\,,$$

$$\mathbf{n}(y_1) = \mathbf{n}(y_3) = \tfrac{l}{4},\quad \mathbf{n}(y_2) = 1 \text{ for } T(\mathbf{D}_l)\,,$$

$$\mathbf{n}(z) = \begin{cases} \tfrac{4}{3} & \text{for } z \in T(\mathbf{E}_6),\ z \neq 0\,, \\ \tfrac{3}{2} & \text{for } z \in T(\mathbf{E}_7),\ z \neq 0\,. \end{cases}$$

Thus, the norm of any element in the discriminant $T(R)$ of a root system R can easily be calculated. Of course, in general $\mathbf{n}(s + s') \neq \mathbf{n}(s) + \mathbf{n}(s')$ for $s, s' \in T(R)$. However, $\mathbf{n}(s + s') = \mathbf{n}(s) + \mathbf{n}(s')$ holds true if s, s' belong to the discriminants $T(R_1)$, $T(R_2)$ of mutually orthogonal root sub-systems.

Only the weights of admissible elements may occur with non-vanishing coefficient in the weight enumerator polynomial P_M of a putative (admissible) metabolizer M.

Before embarking on using the duality theorem, it is possible, in some favorable cases, to eliminate a root system directly by inspection:

If $M \subset T(R)$ is an admissible metabolizer, then for every prime number p, the p-component M_p of M is an admissible metabolizer for the induced bilinear form on the p-component $T(R)_p$ of $T(R)$. There are cases of root systems R and suitable choice of p for which it is apparent that no metabolizer of $T(R)_p$ is admissible. As an example, suppose that $R = \mathbf{A}_2 \boxplus \mathbf{A}_5 \boxplus R'$, where the order of $T(R')$ is prime to 3. Then, $T(R)_3 = T(\mathbf{A}_2 \boxplus \mathbf{A}_5)_3 = T(\mathbf{A}_2) \boxplus T(\mathbf{A}_5)_3 = \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$ generated by $s_1 = (1,0)$, $s_2 = (0,2)$, where $(1,0)$ stands for the projection of $x_1 \in (\mathbf{ZA}_2)^{\#}$ in $T(\mathbf{A}_2) \boxplus T(\mathbf{A}_5)_3$ and $(0,2)$ stands for the projection of $x_2 \in (\mathbf{ZA}_5)^{\#}$ in $T(\mathbf{A}_2) \boxplus T(\mathbf{A}_5)_3$ in the notations of Section 3. Now, $\mathbf{n}(s_1) = \frac{2}{3}$ and $\mathbf{n}(s_2) = \frac{4}{3}$, and for every $s \in T(\mathbf{A}_2 \boxplus \mathbf{A}_5)_3$ one has $\mathbf{n}(s) \leqslant 2$.

This argument eliminates the root systems of the form $R = X \boxplus R'$, with $T(R')$ of order prime to 3 if X is any member of the following (small but frequently arising) black list:

$$X = \mathbf{A}_2 \boxplus \mathbf{A}_5, \quad \mathbf{2A}_2 \boxplus \mathbf{2A}_5, \quad \mathbf{2A}_2 \boxplus \mathbf{A}_5 \boxplus \mathbf{E}_6 .$$

Similarly, $R = \mathbf{mA}_2 \boxplus \mathbf{nA}_5 \boxplus \mathbf{A}_8 \boxplus R'$, with $T(R')$ of order prime to 3 cannot occur for any $m, n \geqslant 0$.

Indeed, for any putative admissible metabolizer M, one should have $M_3 \subset T(\mathbf{mA}_2 \boxplus \mathbf{nA}_5)_3 \boxplus 3T(\mathbf{A}_8)$ because any $s \in M_3$ with $3s \neq 0$ would produce an element $s' = 3s = (0^m, 0^n, \pm 3) \in M_3$, $s' \neq 0$, of norm $\mathbf{n}(s') = 2$, which is inacceptable.

But then $M_3' = M_3 \cap T(\mathbf{mA}_2 \boxplus \mathbf{nA}_5)_3$ would be a metabolizer in $T(\mathbf{mA}_2 \boxplus \mathbf{nA}_5)_3$, and therefore $M_0 = M \cap T(R_0)$ a metabolizer in $T(R_0)$, where $R_0 = \mathbf{mA}_2 \boxplus \mathbf{nA}_5 \boxplus R'$. (The subgroup M_3' is obviously self-orthogonal and it has the right order.) Setting $\pi_0 : (\mathbf{ZR_0})^{\#} \to T(R_0)$, the natural projection, the inverse image $L_0 = \pi_0^{-1}(M_0)$ would be a unimodular sublattice and hence an orthogonal summand of L.

If no such simple argument is available, the root system is to be tested using the duality theorem of Section 4.

For a given root system R, the coefficients in P_M of weight monomials which are not representable by any admissible elements in $T(R)$ must be 0. The duality theorem, using $M = M^{\perp}$, is then a linear system for the remaining coefficients of P_M which must be solvable in non-negative integers. In many cases, this system is not even solvable in rational numbers or if it is, some coefficients turn out to be negative or fractional. Here, all cases occur. In most of the remaining cases where the existence of the polynomial is not prohibited by MacWilliams duality, an admissible metabolizer and hence an even unimodular lattice can actually be constructed.

Completeness of the lists thus relies on a lengthy elimination procedure, let alone the heavy use of machine testing, subject to all sorts of failure. It would certainly be desirable to supply an alternate, perhaps less computational, approach.

The above classification program really begins with the root systems of deficiency 8. For the root systems of deficiency 0, there is another, fairly different method, due to H. Koch and B. Venkov, which we recall in the next paragraph.

NOTATIONS IN THE TABLES

The notation for root systems R is self-explanatory: If e.g. $R = \mathbf{8A}_1 \boxplus \mathbf{8A}_3$, then $\mathbf{Z}R$ is the orthogonal direct sum

$$\mathbf{Z}R = \mathbf{ZA}_1 \boxplus \cdots \boxplus \mathbf{ZA}_1 \boxplus \mathbf{ZA}_3 \boxplus \cdots \boxplus \mathbf{ZA}_3$$

of 8 copies of $\mathbf{ZA}_1$ and 8 copies of $\mathbf{ZA}_3$.

In order to describe a unimodular lattice L containing $\mathbf{Z}R$ we display a *filling set* S, i.e. a set of vectors in $(\mathbf{Z}R)^\#$ which together with $\mathbf{Z}R$ generate L. The terminology is intended to be reminiscent of the similar notion of a glueing set occuring in the paper of J. Conway and V. Pless [CP].

Let $R = R_1 \boxplus \cdots \boxplus R_r$ be the decomposition of R in irreducible components. The vectors in the filling set S contained in

$$(\mathbf{Z}R)^\# = (\mathbf{Z}R_1)^\# \boxplus \cdots \boxplus (\mathbf{Z}R_r)^\#$$

are specified by their coordinates in the successive $(\mathbf{Z}R_i)^\#$, $i = 1, \ldots, r$.

Vectors in the filling set are taken with minimal norm in their class modulo $\mathbf{Z}R$. It is thus easy to read off the norm of an element in S from its displayed expression in coordinates. If the i-th irreducible component R_i of R is $\mathbf{A}_l, \mathbf{D}_l, \mathbf{E}_6$, or $\mathbf{E}_7$, the number k as the i-th coordinate of a vector of S stands for the element noted $x_k(R_i)$ in Section 3.

In order (hopefully) to improve readability, I have separated by a semi-colon the components of a filling vector belonging to different multiple root systems. Thus, for instance $s = (1; 2; 1, 0)$ in the filling set for the root system $\mathbf{A}_3 \boxplus \mathbf{A}_{15} \boxplus \mathbf{2E}_7$, the 16-nth root system with deficiency 8 occuring in the tables, stands for the vector $s = x_1(\mathbf{A}_3) + x_2(\mathbf{A}_{15}) + x_1(\mathbf{E}_7) + 0$ in $(\mathbf{ZA}_3)^\# \boxplus (\mathbf{ZA}_{15})^\# \boxplus (\mathbf{ZE}_7)^\# \boxplus (\mathbf{ZE}_7)^\#$. Its norm is $\frac{1 \cdot 3}{4} + \frac{2 \cdot 14}{16} + \frac{3}{2} + 0 = 4$.

After the filling set, the reader will find the weight enumerator polynomial, sometimes just called the "polynomial" of the metabolizer $M = \pi(L)$, where $\pi : (\mathbf{Z}R)^\# \to T(R)$. The weights refer to the indicated decomposition of the root system under discussion, i.e. one variable only for each multiple factor nR_i, where R_i is irreducible. Thus, for instance, the term $56x^4y^2$ in the polynomial for $R = \mathbf{8A}_1 \boxplus \mathbf{8A}_3$ means that the metabolizer M contains

56 vectors with 4 non-zero coordinates among the first 8 corresponding to $T(\mathbf{A}_1)^8$ and 2 non-zero coordinates among the last 8 corresponding to $T(\mathbf{A}_3)^8$. As an example, we find among these vectors the images in $T(R)$ of the vectors s_4, s_5, s_6, s_7 of the filling set.

The root systems with a fixed deficiency are listed in alphabetical order.

1. Root systems with deficiency 0

This case has been treated by H. Koch and B. Venkov. (See [KV], Satz 3.) If L is an even unimodular lattice of rank 32 with a complete root system of deficiency 0, then L contains 32 mutually orthogonal vectors of scalar square 2, i.e. $a_1, \ldots, a_{32} \in L$ such that $(a_i, a_j) = 2\delta_{ij}$.

Let $N = \mathbf{Z}a_1 \boxplus \mathbf{Z}a_2 \boxplus \ldots \boxplus \mathbf{Z}a_{32}$ and let $N^{\#} = \mathbf{Z}\alpha_1 \boxplus \ldots \boxplus \mathbf{Z}\alpha_{32}$ be the dual lattice, where $\alpha_i = \frac{1}{2} a_i$.

Since $(x, u) \in \mathbf{Z}$ for all $x \in L$, $u \in N$, we have $L \subset N^{\#}$. The quotient $N^{\#}/N$ is the 32-dimensional vector space $\mathbf{F}_2^{32}$ with the standard scalar product $(\varepsilon_i, \varepsilon_j) = \frac{1}{2}\delta_{ij}$ (induced by the scalar product on $N^{\#}$), where ε_i stands for the image of α_i under the projection $\pi : N^{\#} \to N^{\#}/N$.

The image $C_L = \pi(L)$ of the lattice L is then a self-dual code (of dimension 16) in $\mathbf{F}_2^{32}$. Because L is even, it follows that C_L is a doubly-even code (i.e. all code words have a weight divisible by 4).

Now, the doubly-even self-dual codes in $\mathbf{F}_2^{32}$ have been classified by J. Conway and V. Pless in [CP]. There are 85 of them. Crossing out from this list the decomposable ones, we arrive at a list of 75 codes, and therefore 75 irreducible even unimodular lattices, corresponding to 62 root systems.

For the details, see [CP] and [KV].

It turns out that all the examples of non-isomorphic even unimodular 32-dimensional lattices with the same complete root system occur in the case of deficiency 0.

The reader who wishes to see these examples explicitly must therefore turn to [CP].

In the following subsections 2 to 6, containing the list of lattices with non-zero deficiency, each realizable root system uniquely determines the lattice to which it belongs.

2. Root systems with deficiency 8

There are 29 even unimodular lattices of rank 32 having a complete root system of deficiency 8. Each lattice is uniquely determined by its root system.

(1) $8A_1 \boxplus 8A_3$

A filling set for the corresponding lattice consists of the following 8 vectors

$$\begin{aligned}
s_0 &= (0, 0, 0, 0, 0, 0, 0, 0; 1, 1, 1, 1, 1, 1, 1, 1),\\
s_1 &= (1, 1, 0, 0, 0, 0, 0, 0; 0, 0, 0, 0, 1, 1, 1, 1),\\
s_2 &= (0, 1, 1, 0, 0, 0, 0, 0; 0, 0, 1, 1, 1, 1, 0, 0),\\
s_3 &= (0, 0, 0, 1, 1, 0, 0, 0; 0, 1, 3, 0, 0, 1, 3, 0),\\
s_4 &= (1, 1, 1, 1, 0, 0, 0, 0; 0, 0, 0, 0, 2, 2, 0, 0),\\
s_5 &= (0, 0, 1, 1, 1, 1, 0, 0; 0, 0, 0, 0, 0, 2, 2, 0),\\
s_6 &= (0, 1, 1, 0, 0, 1, 1, 0; 0, 0, 2, 0, 0, 2, 0, 0),\\
s_7 &= (0, 0, 0, 0, 1, 1, 1, 1; 0, 0, 0, 0, 0, 0, 2, 2).
\end{aligned}$$

The weight enumerator polynomial is

$$\begin{aligned}
P(x, y) = 1 &+ x^8 + 56x^4y^2 + 14y^4 + 112x^2y^4 + 112x^4y^4\\
&+ 112x^6y^4 + 14x^8y^4 + 896x^4y^5 + 672x^2y^6 + 56x^4y^6\\
&+ 672x^6y^6 + 896x^4y^7 + 17y^8 + 112x^2y^8 + 224x^4y^8\\
&+ 112x^6y^8 + 17x^8y^8.
\end{aligned}$$

The (rather delicate) discussion of this root system in presented in Section 7.

(2) $4A_1 \boxplus 4A_5 \boxplus D_8$

Filling set $S = \langle s_1, s_2, s_3, s_4, s_5, s_6, s_7 \rangle$, where

$$\begin{aligned}
s_1 &= (1, 0, 0, 0; 3, 0, 0, 0; 1), & s_2 &= (0, 1, 0, 0; 0, 3, 0, 0; 1),\\
s_3 &= (0, 0, 1, 0; 0, 0, 3, 0; 1), & s_4 &= (0, 0, 0, 1; 0, 0, 0, 3; 1),\\
s_5 &= (1, 1, 1, 1; 0, 0, 0, 0; 3), & s_6 &= (0, 0, 0, 0; 0, 2, 2, 2; 0),\\
s_7 &= (0, 0, 0, 0; 2, 0, 2, 4; 0).
\end{aligned}$$

Polynomial

$$\begin{aligned}
P(x, y, z) = 1 &+ 6x^2y^2 + 8y^3 + 24x^2y^3 + 24x^2y^4 + 9x^4y^4 + x^4z + 4xyz\\
&+ 4x^3yz + 6x^2y^2z + 36xy^3z + 24x^2y^3z + 36x^3y^3z\\
&+ 8x^4y^3z + 9y^4z + 32xy^4z + 24x^2y^4z + 32x^3y^4z.
\end{aligned}$$

(3) $2A_1 \boxplus 2A_3 \boxplus 2A_7 \boxplus D_{10}$

Filling set

$S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$\begin{aligned}
s_1 &= (1, 0; 2, 0; 0, 0; 1), & s_2 &= (0, 1; 0, 2; 0, 0; 3),\\
s_3 &= (0, 0; 1, 1; 2, 0; 2), & s_4 &= (1, 1; 0, 1; 1, 1; 1).
\end{aligned}$$

Polynomial

$$P(x, y, z, t) = 1 + 2y^2z + 4x^2y^2z + z^2 + 4yz^2 + 8xyz^2 + 8xy^2z^2 + 4x^2y^2z^2 + 2xyt + x^2y^2t + 2x^2zt + 4xyzt + 4y^2zt + 8xy^2zt + 4xz^2t + 8yz^2t + 10xyz^2t + 12x^2yz^2t + 12y^2z^2t + 20xy^2z^2t + 9x^2y^2z^2t.$$

(4) $$\mathbf{2A_1 \boxplus 2A_9 \boxplus D_{12}}$$

Filling set

$$S = <(1, 0; 5, 0; 1), (0, 1; 0, 5; 1), (1, 0; 0, 5; 2), (0, 0; 2, 4; 0)>.$$

Polynomial

$$P(x, y, z) = 1 + 4y^2 + 5x^2y^2 + x^2z + 4xyz + 5y^2z + 16xy^2z + 4x^2y^2z.$$

(5) $$\mathbf{A_1 \boxplus A_3 \boxplus 2A_7 \boxplus D_7 \boxplus E_7}$$

Filling set $S = < s_1, s_2, s_3 >$, where

$$s_1 = (1; 1; 1, 3; 0; 0), \quad s_2 = (0; 1; 2, 4; 1; 0), \quad s_3 = (1; 0; 0, 4; 0; 1).$$

Polynomial

$$P(x, y, z, t, u) = 1 + z^2 + 2yz^2 + 4xyz^2 + 6yzt + 2z^2t + 4xz^2t + 4yz^2t + 8xyz^2t + 2xzu + 4yz^2u + 2xyz^2u + xytu + 4xyztu + 4z^2tu + 2xz^2tu + 8yz^2tu + 5xyz^2tu.$$

(6) $$\mathbf{A_1 \boxplus A_5 \boxplus A_{11} \boxplus D_5 \boxplus D_{10}}$$

Filling set $S = < (1;3;0;2;2), (0;3;0;0;1), (1;0;3;1;0), (0;2;4;0;0) >$.
Polynomial

$$P(x, y, z, t, u) = 1 + 2yz + zt + 2xzt + 2yzt + 4xyzt + yu + xzu + 2yzu + 5xyzu + xtu + xytu + 2ztu + 13yztu + 10xyztu.$$

(7) $$\mathbf{A_1 \boxplus A_{17} \boxplus D_{14}}$$

Filling set $S = < (1; 0; 1), (0; 9; 3), (0; 6; 0) >$.
Polynomial $P(x, y, z) = 1 + 2y + xz + 3yz + 5xyz.$

(8) $$\mathbf{8A_3 \boxplus 2D_4}$$

Filling set $S = < s_1, s_2, s_3, s_4, s_5, s_6 >$, where

$s_1 = (1, 1, 1, 1, 1, 1, 1, 1; 0, 0)$, $s_2 = (0, 1, 0, 0, 1, 3, 2, 1; 0, 0)$,
$s_3 = (0, 0, 1, 0, 1, 0, 1. 1; 1, 0)$, $s_4 = (0, 0, 0, 1, 0, 1, 3, 3; 2, 0)$,
$s_5 = (0, 2, 0, 0, 0, 2, 0, 0; 1, 1)$, $s_6 = (0, 2, 0, 0, 2, 0, 0, 0; 2, 2)$.

Polynomial

$$P(x, y) = 1 + 14x^4 + 16x^5 + 16x^7 + 17x^8 + 48x^4y + 288x^6y + 48x^8y + 12x^2y^2 + 24x^4y^2 + 240x^5y^2 + 12x^6y^2 + 240x^7y^2 + 48x^8y^2.$$

(9) $\mathbf{8A_3 \boxplus D_8}$

Filling set $S = < s_1, s_2, s_3, s_4, s_5 >$, where

$s_1 = (1, 0, 0, 0, 2, 1, 1, 1; 0)$, $s_2 = (0, 1, 0, 2, 1, 0, 1, 1; 0)$,
$s_3 = (0, 0, 1, 1, 0, 2, 1, 1; 0)$, $s_4 = (0, 0, 0, 1, 1, 1, 3, 2; 1)$,
$s_5 = (0, 0, 0, 0, 0, 0, 2, 2; 3)$.

Polynomial

$$P(x, y) = 1 + 14x^4 + 48x^5 + 48x^7 + 17x^8 + 4x^2y + 24x^4y + 112x^5y + 100x^6y + 112x^7y + 32x^8y.$$

(10) $\mathbf{7A_3 \boxplus D_{11}}$

Filling set $S = < s_1, s_2, s_3, s_4 >$, where

$s_1 = (1, 0, 0, 2, 1, 1, 1; 0)$, $s_2 = (0, 1, 0, 1, 2, 3, 1; 0)$,
$s_3 = (0, 0, 1, 3, 3, 2, 1; 0)$, $s_4 = (0, 0, 0, 1, 3, 1, 2; 1)$.

Polynomial

$$P(x, y) = 1 + 7x^4 + 42x^5 + 14x^7 + 7x^3y + 70x^4y + 98x^6y + 17x^7y.$$

(11) $\mathbf{6A_3 \boxplus 2D_7}$

Filling set $S = < s_1, s_2, s_3, s_4 >$, where

$s_1 = (1, 0, 0, 0, 1, 1; 0, 1)$, $s_2 = (0, 1, 0, 0, 1, 3; 1, 0)$,
$s_3 = (0, 0, 1, 0, 1, 2; 1, 1)$, $s_4 = (0, 0, 0, 1, 2, 1; 3, 1)$.

Polynomial

$$P(x, y) = 1 + 3x^4 + 12x^5 + 24x^3y + 12x^4y + 48x^5y + 12x^6y + 3x^2y^2 + 24x^3y^2 + 48x^4y^2 + 36x^5y^2 + 33x^6y^2.$$

(12) $\mathbf{4A_3 \boxplus 4D_5}$

Filling set $S = <s_1, s_2, s_3, s_4>$, where

$$s_1 = (1, 1, 1, 1; 2, 0, 0, 0), \quad s_2 = (1, 1, 0, 0; 1, 1, 0, 0),$$
$$s_3 = (0, 1, 3, 0; 0, 1, 1, 0), \quad s_4 = (0, 0, 3, 1; 0, 0, 1, 1).$$

Polynomial

$$P(x, y) = 1 + x^4 + 8x^4y + 36x^2y^2 + 24x^4y^2 + 96x^3y^3 + 8x^4y^3 + y^4 + 8xy^4 + 24x^2y^4 + 8x^3y^4 + 41x^4y^4.$$

(13) $\mathbf{2A_3 \boxplus 2A_7 \boxplus 2D_6}$

Filling set

$$S = <(1,0;1,1;1,0), (1,1;2,0;2,0), (2,0;0,0;1,1), (0,2;0,0;3,3)>.$$

Polynomial

$$P(x, y, z) = 1 + 2x^2y + y^2 + 4xy^2 + 8x^2yz + 16xy^2z + 24x^2y^2z + 2xz^2 + x^2z^2 + 2yz^2 + 4xyz^2 + 8x^2yz^2 + 4y^2z^2 + 22xy^2z^2 + 29x^2y^2z^2.$$

(14) $\mathbf{A_3 \boxplus A_5 \boxplus A_{11} \boxplus D_6 \boxplus E_7}$

Filling set $S = <s_0, s_1, s_2, s_3>$, where

$$s_0 = (1; 3; 3; 0; 1), \quad s_1 = (2; 3; 0; 1; 0),$$
$$s_2 = (0; 0; 6; 3; 1), \quad s_3 = (0; 2; 4; 0; 0).$$

Polynomial

$$P(x, y, z, t, u) = 1 + xz + 2yz + 2xyz + xyt + 2xzt + 3yzt + 12xyzt + 6xyzu + xtu + ytu + ztu + 2xztu + 4yztu + 9xyztu.$$

(15) $\mathbf{A_3 \boxplus A_{11} \boxplus D_{12} \boxplus E_6}$

Filling set

$$S = <(1; 3; 2; 0), (0; 6; 1; 0), (0; 4; 0; 1)>.$$

Polynomial

$$P(x, y, z, t) = 1 + xy + xz + yz + 4xyz + 2yt + 2xyt + 2yzt + 10xyzt.$$

(16) $\mathbf{A_3 \boxplus A_{15} \boxplus 2E_7}$

Filling set $S = \langle (1;\ 2;\ 1,\ 0),\ (2;\ 0;\ 1,\ 1) \rangle$.
Weight polynomial

$P(x, y, z) = 1 + y + 2xy + 8xyz + xz^2 + 2yz^2 + xyz^2$.

(17) $\mathbf{4A_5 \boxplus 2D_6}$

Filling set $S = \langle S_2,\ S_3 \rangle$, where

$$S_2 = \langle (3,0,0,0;1,2),\ (0,3,0,0;3,2),\ (0,0,3,0;2,1),\ (0,0,0,3;2,3) \rangle,$$
$$S_3 = \langle (0,2,2,2;0,0),\ (2,0,2,4;0,0) \rangle.$$

Polynomial

$$P(x, y) = 1 + 8x^3 + 2x^2y + 20x^3y + 32x^4y + 4xy^2 + 4x^2y^2 + 40x^3y^2 + 33x^4y^2.$$

(18) $\mathbf{4A_5 \boxplus D_{12}}$

Filling set $S = \langle S_2,\ S_3 \rangle$, where

$$S_2 = \langle (3,\ 3,\ 3,\ 3;\ 0),\ (3,\ 3,\ 0,\ 0;\ 1),\ (0,\ 3,\ 3,\ 0;\ 2) \rangle$$
$$S_3 = \langle (0,\ 2,\ 2,\ 2;\ 0),\ (2,\ 0,\ 2,\ 4;\ 0) \rangle.$$

Polynomial

$P(x, y) = 1 + 8x^3 + 9x^4 + 6x^2y + 24x^3y + 24x^4y$.

(19) $\mathbf{3A_5 \boxplus D_4 \boxplus E_6 \boxplus E_7}$

Filling set $S = \langle s_1,\ s_2,\ s_3,\ s_4,\ s_5 \rangle$, where

$$s_1 = (0,\ 3,\ 3;\ 1;\ 0;\ 0), \qquad s_2 = (3,\ 0,\ 3;\ 2;\ 0;\ 0),$$
$$s_3 = (3,\ 3,\ 3;\ 0;\ 0;\ 1), \qquad s_4 = (2,\ 2,\ 0;\ 0;\ 1;\ 0),$$
$$s_5 = (2,\ 4,\ 2;\ 0;\ 0;\ 0).$$

Polynomial

$$P(x, y, z, t) = 1 + 2x^3 + 3x^2y + 6x^3y + 6x^2z + 6x^2yz + 12x^3yz + 3x^3t + 3xyt + 6x^3yt + 6x^3zt + 12x^2yzt + 6x^3yzt.$$

(20) $\mathbf{2A_5 \boxplus D_{10} \boxplus 2E_6}$

Filling set

$$S = \langle (3,\ 0;\ 1;\ 0,\ 0),\ (0,\ 3;\ 3;\ 0,\ 0),\ (2,\ 2;\ 0;\ 1,\ 0),\ (2,\ 4;\ 0;\ 0,\ 1) \rangle.$$

Polynomial

$$P(x, y, z) = 1 + 2xy + x^2y + 4x^2z + 12x^2yz + 4xz^2 + 4xyz^2 + 8x^2yz^2.$$

(21) $\mathbf{A_5 \boxplus A_{11} \boxplus D_9 \boxplus E_7}$

Filling set

$$S = <(0; 3; 1; 1), (3; 6; 0; 1), (2; 4; 0; 0)>.$$

Polynomial

$$P(x, y, z, t) = 1 + 2xy + yz + 8xyz + 3xyt + xzt + 2yzt + 6xyzt.$$

(22) $\mathbf{2A_7 \boxplus 2D_5 \boxplus D_8}$

Filling set $S = <(1, 1; 1, 0; 2), (2, 0; 1, 1; 0), (0, 0; 2, 2; 1)>$.
Polynomial

$$P(x, y, z) = 1 + x^2 + 4x^2y + 6xy^2 + 4x^2y^2 + 2xz + 20x^2yz + y^2z + 4xy^2z + 21x^2y^2z.$$

(23) $\mathbf{2A_7 \boxplus D_5 \boxplus D_{13}}$

Filling set $S = <(1, 3; 1; 0), (2, 0; 1, 1)>$.
Polynomial $P(x, y, z) = 1 + x^2 + 6x^2y + 6x^2z + 6xyz + 12x^2yz$.

(24) $\mathbf{2A_7 \boxplus 2D_9}$

Filling set $S = <(1, 1; 1, 0), (2, 0; 1, 1)>$.
Polynomial $P(x, y) = 1 + x^2 + 12x^2y + 6xy^2 + 12x^2y^2$.

(25) $\mathbf{2A_9 \boxplus D_{14}}$

Filling set $S = <(5, 0; 1), (0, 5; 3), (2, 4; 0)>$.
Polynomial $P(x, y) = 1 + 4x^2 + 2xy + 13x^2y$.

(26) $\mathbf{2A_9 \boxplus 2E_7}$

Filling set $S = <(5, 0; 1, 0), (0, 5; 0, 1), (2, 4; 0, 0)>$.
Weight polynomial $P(x, y) = 1 + 4x^2 + 2xy + 8x^2y + 5x^2y^2$.

(27) $\mathbf{A_{11} \boxplus D_{15} \boxplus E_6}$

Filling set $S = <(3; 1; 0), (4; 0; 1)>$.
Polynomial $P(x, y, z) = 1 + 3xy + 2xz + 6xyz$.

(28) $\mathbf{A}_{15} \boxplus \mathbf{D}_5 \boxplus \mathbf{D}_{12}$

Filling set $S = \langle (2;\ 1;\ 1),\ (0;\ 2;\ 3) \rangle$.
Polynomial $P(x, y, z) = 1 + x + 2xy + 2xz + yz + 9xyz$.

(29) $\mathbf{A}_{15} \boxplus \mathbf{D}_{17}$

Filling set $S = \langle (2;\ 1) \rangle$.
Polynomial $P(x, y) = 1 + x + 6xy$.

3. Root systems with deficiency 12

There are 10 root systems of rank 32 and deficiency 12 appearing as the root system of a (unique) even unimodular lattice of rank 32.

(1) $4\mathbf{A}_1 \boxplus 4\mathbf{A}_7$

The filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$ is given by

$$s_1 = (1,\ 0,\ 0,\ 0;\ 1,\ 1,\ 1,\ 1), \quad s_2 = (1,\ 1,\ 0,\ 0;\ 2,\ 2,\ 0,\ 0),$$
$$s_3 = (0,\ 1,\ 1,\ 0;\ 0,\ 2,\ 6,\ 0), \quad s_4 = (1,\ 1,\ 1,\ 1;\ 0,\ 0,\ 0,\ 4).$$

The weight enumerator polynomial of the corresponding metabolizer reads

$$P(x, y) = 1 + 4x^4y + 6y^2 + 24x^2y^2 + 48x^2y^3 + 4x^4y^3 + 9y^4 + 64xy^4 + 24x^2y^4 + 64x^3y^4 + 8x^4y^4.$$

(2) $4\mathbf{A}_2 \boxplus 4\mathbf{A}_5 \boxplus \mathbf{D}_4$

Filling set $S = \langle s_1, s_2, s_3 \rangle \times \langle s_4, s_5, s_6, s_7 \rangle$, where

$$s_1 = (0,\ 0,\ 0,\ 0; 3,\ 3,\ 3,\ 3;\ 0), \quad s_2 = (0,\ 0,\ 0,\ 0;\ 3,\ 3,\ 0,\ 0;\ 1),$$
$$s_3 = (0,\ 0,\ 0,\ 0;\ 0,\ 3,\ 3,\ 0;\ 2),$$
$$s_4 = (1,\ 1,\ 1,\ 1;\ 2,\ 0,\ 0,\ 0;\ 0),$$
$$s_5 = (1,\ -1,\ 1,\ -1;\ 0,\ 2,\ 0,\ 0;\ 0),$$
$$s_6 = (1,\ 1,\ -1,\ -1;\ 0,\ 0,\ 2,\ 0;\ 0),$$
$$s_7 = (1,\ -1,\ -1,\ 1;\ 0,\ 0,\ 0,\ 2;\ 0).$$

Weight enumerator polynomial

$$P(x, y, z) = 1 + 8x^4y + 24x^2y^2 + 32x^3y^3 + y^4 + 16xy^4 + 24x^2y^4 + 32x^3y^4 + 24x^4y^4 + 6y^2z + 24x^2y^2z + 24x^4y^2z + 96x^2y^3z + 96x^3y^3z + 24x^4y^3z + 48xy^4z + 24x^2y^4z + 96x^3y^4z + 48x^4y^4z.$$

(3) **$4A_2 \boxplus 4E_6$**

Filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$s_1 = (1,0,0,0;1,\ 1,\ 1,\ 1),$$
$$s_2 = (0,1,0,0;1,-1,\ 1,-1),$$
$$s_3 = (0,0,1,0;1,\ 1,-1,-1),$$
$$s_4 = (0,0,0,1;1,-1,-1,\ 1).$$

Weight enumerator polynomial

$$P(x,y) = 1 + 8x^4y + 24x^2y^2 + 32x^3y^3 + 8xy^4 + 8x^4y^4.$$

(4) **$2A_2 \boxplus 2A_{11} \boxplus D_6$**

Filling set

$$S = \langle (0,\ 0;\ 3,\ 3;\ 1),\ (0,\ 0;\ 6,\ 0;\ 2),\ (1,\ 1,\ 4,\ 0;\ 0),\ (1,\ 2;\ 0,\ 4;\ 0) \rangle .$$

Polynomial

$$P(x,y,z) = 1 + 4x^2y + y^2 + 8xy^2 + 4x^2y^2 + 2yz + 4x^2yz + 4y^2z + 24xy^2z + 20x^2y^2z.$$

(5) **$A_2 \boxplus A_9 \boxplus A_{14} \boxplus E_7$**

Filling set $S = \langle (1;\ 0;\ 5;\ 0),\ (0;\ 2;\ 3;\ 0),\ (0;\ 5;\ 0;\ 1) \rangle$.
Weight polynomial

$$P(x,y,z,t) = 1 + 2xz + 4yz + 8xyz + yt + 4yzt + 10xyzt.$$

(6) **$A_2 \boxplus A_{23} \boxplus E_7$**

Filling set $S = \langle (1;\ 8;\ 0),\ (0;\ 6;\ 1) \rangle$.
Weight enumerator polynomial

$$P(x,y,z) = 1 + y + 4xy + 2yz + 4xyz.$$

(7) **$6A_3 \boxplus 2A_7$**

Filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$, where

$$s_1 = (2,\ 1,\ 1,\ 1,\ 1,\ 0;\ 0,\ 0), \qquad s_2 = (1,\ 2,\ 1,\ 3,\ 0,\ 1;\ 0,\ 0),$$
$$s_3 = (1,\ 1,\ 1,\ 0,\ 0,\ 0;\ 1,\ 1), \qquad s_4 = (0,\ 2,\ 1,\ 1,\ 0,\ 0;\ 2,\ 0).$$

Weight enumerator polynomial

$$P(x,y) = 1 + 3x^4 + 12x^5 + 6x^2y + 24x^3y + 48x^5y + 18x^6y + y^2 + 72x^3y^2 + 123x^4y^2 + 132x^5y^2 + 72x^6y^2.$$

(8) $\mathbf{2A_4 \boxplus 2A_9 \boxplus D_6}$

Filling set

$$S = <(0, 0; 5, 0; 1), (0, 0; 0, 5; 3), (1, 0; 2, 2; 0), (0, 1; 2, 8; 0)>.$$

Polynomial

$$P(x, y, z) = 1 + 8x^2y + 8xy^2 + 8x^2y^2 + 2yz + 8x^2yz + y^2z + 24xy^2z + 40x^2y^2z.$$

(9) $\mathbf{A_4 \boxplus A_{19} \boxplus D_9}$

Filling set $S = <(0; 5; 1), (1; 4; 0)>$.
Polynomial $P(x, y, z) = 1 + 4xy + 3yz + 12xyz$.

(10) $\mathbf{A_8 \boxplus A_{17} \boxplus E_7}$

Filling set $S = <(4; 2; 0), (0; 9; 1)>$.
Weight polynomial $P(x, y, z) = 1 + 8xy + yz + 8xyz$.

4. Root systems of deficiency 14

There are 5 root systems with deficiency 14 which appear as a complete root system in an even unimodular lattice of rank 32. There is only one lattice for each realizable root system.

(1) $\mathbf{2A_1 \boxplus 2A_{15}}$

Filling set

$$S = <(1, 0; 2, 2), (1, 1; 4, 0)>.$$

The weight enumerator polynomial is

$$P(x, y) = 1 + 2y + 4x^2y + 5y^2 + 16xy^2 + 4x^2y^2.$$

(2) $\mathbf{10A_2 \boxplus 2E_6}$

A filling set $S = <s_1, s_2, s_3, s_4, s_5, s_6>$ is as follows

$$s_1 = (0, 1, 1, 1, 1, 1, 1, 1, 1, 1; 0, 0),$$
$$s_2 = (1, 1, 0, 1, 1, 2, 2, 2, 2, 1; 0, 0),$$
$$s_3 = (1, 2, 1, 2, 0, 1, 1, 2, 2, 1; 0, 0),$$
$$s_4 = (1, 1, 2, 2, 1, 1, 0, 2, 1, 2; 0, 0),$$
$$s_5 = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0; 1, 0),$$
$$s_6 = (0, 0, 1, 2, 2, 1, 0, 0, 0, 0; 0, 1).$$

The weight enumerator of the corresponding metabolizer is

$$P(x, y) = 1 + 60x^6 + 20x^9 + 60x^4y + 240x^7y + 24x^{10}y + 144x^5y^2 + 180x^8y^2.$$

See the following Section 7 for the relationship of this root system with conference matrices.

(3) $\mathbf{2A_2 \boxplus 2A_3 \boxplus 2A_{11}}$

Filling set

$S = \langle (0,0;1,2;3,0), (0,0;2,1;0,3), (1,1;0,0;4,0), (1,2;0,0;0,4) \rangle.$

Polynomial

$$P(x, y, z) = 1 + 4x^2z + 2yz + 4x^2yz + 4y^2z + 8x^2y^2z + 4xz^2 + 4yz^2 + 24xyz^2 + 20x^2yz^2 + 5y^2z^2 + 36xy^2z^2 + 28x^2y^2z^2.$$

(4) $\mathbf{2A_5 \boxplus 2A_{11}}$

Filling set

$S = \langle (3, 0; 3, 3), (3, 3; 6, 0), (2, 0; 4, 0), (0, 2; 0, 4) \rangle.$

Polynomial

$$P(x, y) = 1 + 4xy + 6x^2y + y^2 + 16xy^2 + 44x^2y^2.$$

(5) $\mathbf{A_{11} \boxplus A_{15} \boxplus E_6}$

Filling set $S = \langle (3; 2; 0), (4; 0; 1) \rangle$.
Polynomial $P(x, y, z) = 1 + y + 6xy + 2xz + 14xyz$.

5. Root systems of deficiency 15

There are 8 root systems of deficiency 15 which occur as the complete root system of an even unimodular lattice of rank 32. Each lattice is uniquely determined by its root system.

(1) $\mathbf{A_1 \boxplus 3A_6 \boxplus A_{13}}$

Filling set

$S = \langle (1; 0, 0, 0; 7), (0; 1, 2, 3; 0), (0; 2, 6, 0; 2) \rangle.$

Polynomial

$$P(x, y, z) = 1 + 6y^3 + xz + 18y^2z + 18xy^2z + 24y^3z + 30xy^3z.$$

Here again, the polynomial is the only candidate satisfying duality. In turn, the given filling set is uniquely determined by the polynomial.

(2) $$\mathbf{A}_1 \boxplus \mathbf{A}_{10} \boxplus \mathbf{A}_{21}$$

Filling set $S = \langle (1;\ 0;\ 11),\ (0;\ 1;\ 8) \rangle$.
Polynomial $P(x, y, z) = 1 + xz + 10yz + 10xyz$.

(3) $$\mathbf{A}_1 \boxplus \mathbf{A}_{31}$$

Filling set $S = \langle (1;\ 4) \rangle$.
Polynomial $P(x, y) = 1 + 3y + 4xy$.

(4) $$13\mathbf{A}_2 \boxplus \mathbf{E}_6$$

Filling set $S = \langle s_0, s_1, s_2, s_3, s_4, s_5, s_6 \rangle$ as follows

$$\begin{aligned}
s_0 &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1;\ 1),\\
s_1 &= (2, 0, 1, 0, 2, 1, 2, 1, 0, 0, 0, 0, 0;\ 0),\\
s_2 &= (0, 2, 0, 1, 0, 2, 1, 2, 1, 0, 0, 0, 0;\ 0),\\
s_3 &= (0, 0, 2, 0, 1, 0, 2, 1, 2, 1, 0, 0, 0;\ 0),\\
s_4 &= (0, 0, 0, 2, 0, 1, 0, 2, 1, 2, 1, 0, 0;\ 0),\\
s_5 &= (0, 0, 0, 0, 2, 0, 1, 0, 2, 1, 2, 1, 0;\ 0),\\
s_6 &= (0, 0, 0, 0, 0, 2, 0, 1, 0, 2, 1, 2, 1;\ 0).
\end{aligned}$$

The weight enumerator is

$$\begin{aligned}
P(x, y) = 1 &+ 156x^6 + 494x^9 + 78x^{12} + 26x^4y + 624x^7y\\
&+ 780x^{10}y + 28x^{13}y.
\end{aligned}$$

Note that $M_0 = M \cap T(13\mathbf{A}_2)$, where M is the metabolizer generated by S in $T(13\mathbf{A}_2 \boxplus \mathbf{E}_6)$, is the cyclic code in $\mathbf{F}_3[\mathbf{x}]/(\mathbf{x}^{13} - \mathbf{1})$ generated by

$$\begin{aligned}
g(x) &= x^7 - x^6 + x^5 - x^4 + x^2 - 1\\
&= (x - 1)(x^3 + x^2 - 1)(x^3 - x^2 - x - 1),
\end{aligned}$$

with roots $\alpha^4, \alpha^7, \alpha^8, \alpha^{10}, \alpha^{11}, \alpha^{12}, \alpha^{13} = 1$, where α is a root of $X^3 - X - 1$ in $\mathbf{F}_{27}$.

(5) $$\mathbf{A}_2 \boxplus \mathbf{A}_5 \boxplus \mathbf{A}_8 \boxplus \mathbf{A}_{17}$$

Filling set

$$S = \langle (0;\ 3;\ 0;\ 9),\ (1;\ 4;\ 1;\ 4),\ (1;\ 2;\ 3;\ 0) \rangle.$$

Polynomial

$P(x, y, z, t) = 1 + 2xyz + yt + 4xyt + 2zt + 6xzt + 14yzt + 24xyzt.$

Here, in order to prove uniqueness, one should first observe that the weight enumerator of the metabolizer is uniquely determined by the duality theorem of Section 4. It is then easy to see that the above filling set is the only possible one.

(6) $$\mathbf{A}_2 \boxplus 3\mathbf{A}_8 \boxplus \mathbf{E}_6$$

Filling set

$$S = <(0;\ 1,\ 1,\ 1;\ 1),\ (1;\ 3,\ 0,\ 0;\ 1),\ (1;\ 0,\ 3,\ 0;\ 1)>.$$

Weight enumerator

$$P(x, y, z) = 1 + 6y^2 + 2y^3 + 18xy^3 + 6xyz + 6xy^2z + 18y^3z + 24xy^3z.$$

For the proof of uniqueness, one first observes that the above polynomial is the only one compatible with the requirement of duality. Then, the only 6 candidates for the weight xyz are $\pm(1; 3, 0, 0; 1)$, $\pm(1; 0, 3, 0; 1)$ and $\pm(1; 0, 0, 3; 1)$.

The vector $(0; 1, 1, 1; 1)$ is then uniquely determined, up to obvious automorphisms, by the requirement of compatibility with the other 3 vectors.

(7) $$\mathbf{A}_6 \boxplus \mathbf{A}_{20} \boxplus \mathbf{E}_6$$

Filling set $S = <(0;\ 7;\ 1),\ (2;\ 3;\ 0)>$.
Polynomial $P(x, y, z) = 1 + 6xy + 2yz + 12xyz$.

(8) $$\mathbf{A}_{26} \boxplus \mathbf{E}_6$$

Filling set $S = <(3;\ 1)>$.
Weight enumerator $P(x, y) = 1 + 2x + 6xy$.

6. Root systems of deficiency 16

There are 5 root systems of deficiency 16 occuring as the root system of even unimodular lattices of rank 32. Each of these lattices is determined by its root system.

(1) $$16\mathbf{A}_2$$

The system of filling vectors can be taken as the rows of an 8×16 matrix

$$S = (I, H),$$

where I is the 8×8 identity matrix and H is the Hadamard matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

The weight enumerator is

$P(x) = 1 + 224x^6 + 2720x^9 + 3360x^{12} + 256x^{15}$.

The uniqueness of the lattice with this root system follows from the classification of self-dual codes in $\mathbf{F}_3^{16}$ due to J. Conway, V. Pless and N. Sloane in [CPS].

(2) $\mathbf{2A_2 \boxplus 2A_{14}}$

Filling set $S = \langle (1, 0; 5, 0), (0, 1; 0, 5), (0, 0; 3, 6) \rangle$.
Weight enumerator $P(x, y) = 1 + 4xy + 4y^2 + 16xy^2 + 20x^2y^2$.

(3) $\mathbf{8A_4}$

Filling set $S = \langle s_1, s_2, s_3, s_4 \rangle$, where s_1, s_2, s_3, s_4 can be taken to be the rows of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

The weight enumerator is

$P(x) = 1 + 48x^4 + 32x^5 + 288x^6 + 128x^7 + 128x^8$.

For the proof of uniqueness, see the comments in the next section.

(4) $\mathbf{4A_8}$

Filling set $S = \langle (1, 1, 4, 0), (1, -1, 0, 4) \rangle$.
Weight enumerator $P(x) = 1 + 32x^3 + 48x^4$.

(5) $\mathbf{2A_{16}}$

Filling set $S = \langle (1, 4) \rangle$.
Weight enumerator $P(x) = 1 + 16x^2$.