2. Convex polyhedra

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There are several reasons why it is better to use several convex building blocks than only one. Firstly, as we have already pointed out, this is necessary if we are to deal with all geometrically finite groups. Secondly many of the most interesting examples are constructed using more than one piece, for example the two ideal regular hyperbolic tetrahedra used to give a complete hyperbolic structure to the figure-eight complement (see [Thu, Thu80]). Thirdly the hypotheses come up naturally in the proof; if one starts with a single convex piece, the natural inductive proof inexorably leads one to consider glueing together several convex pieces in lower dimensions. Fourthly, it may be convenient to use a non-convex fundamental domain, rather than a convex fundamental domain. The non-convex fundamental domains that arise in practice can be cut into a finite number of convex pieces, making our hypotheses applicable.

One way in which our treatment differs from all previous treatments, is that we do not assume we start with an embedded fundamental domain. Instead the fundamental domain is expressed as the union of convex cells, each of which can be separately embedded, without knowing to begin with that their union can be embedded. For example, suppose we are given three planar wedges of angle $5\pi/6$, $6\pi/7$ and $7\pi/8$ with face-pairings glueing them together. The union of these pieces cannot form a fundamental domain, because their union after glueing cannot be embedded. The point here is whether this non-embeddability or embeddability needs to be checked beforehand. Our proof shows that the usual checks for Poincaré's Theorem, in the case where there is only one convex piece, in any case imply the embeddability of the potential fundamental domain, so no special separate check is necessary. In this case the extra necessary checking is easy, but in a more complicated situation, the algorithm presented here could lead to significant saving of time and complication.

2. Convex polyhedra

Let $\mathbf{X}^n$ be hyperbolic, euclidean or spherical n-dimensional space. A hyperplane (that is, a codimension-one $\mathbf{X}$-subspace) divides $\mathbf{X}^n$ into two components; we will call the closure of either of them a *half-space* in $\mathbf{X}^n$. Any $\mathbf{X}$-subspace is the intersection of hyperplanes, and *vice versa*.

Definition 2.1 (convex polyhedron). A connected subset P of $\mathbf{X}^n$ is called a *convex polyhedron* if it is the intersection of a family $\mathscr{H}$ of half-spaces with the property that each point of P has a neighbourhood meeting at most a finite number of boundaries of elements of $\mathscr{H}$. A convex polyhedron in $\mathbf{X}^n$ is said to be *thick* in $\mathbf{X}^n$ if it has non-empty interior.

REMARK 2.2 (antipodal points). In $\mathbf{H}^n$ and $\mathbf{E}^n$ any two points are joined by a unique geodesic segment, so the same property holds in any intersection of half-spaces. In particular intersections of half-spaces are connected. In $\mathbf{S}^n$, we have to make do with a slightly weaker form of this, in which any two points x and y, such that $d(x, y) < \pi$, are joined by a unique shortest geodesic segment, in any intersection of half-spaces. In $\mathbf{S}^n$ a pair of antipodal points can be obtained as the intersection of $n + 1$ half-spaces. Furthermore one can easily check that if an intersection P of half-spaces in $\mathbf{S}^n$ does not enjoy the property that any two points of P are joined by a geodesic arc within P, then P must be a pair of antipodal points. A single point in $\mathbf{S}^n$ is of course an intersection of half-spaces. So the only intersection of a locally finite family of half-spaces which is not a convex polyhedron is a pair of antipodal points in the sphere.

LEMMA 2.3 (interior). *An intersection P of half-spaces in $\mathbf{X}^n$ either has non-empty interior in $\mathbf{X}^n$ or is contained in a hyperplane. Moreover, if the interior of P is not empty, it is dense in P.*

Proof of 2.3. We may suppose that $P \neq \varnothing$. Let $\mathscr{S}$ be the set of non-empty $\mathbf{X}$-subspaces S of $\mathbf{X}^n$ such that $P \cap S$ has non-empty S-interior (V, say) and such that V is dense in $P \cap S$. Clearly $\mathscr{S}$ has a 0-dimensional member, so it is not empty. Let S be a maximal element of $\mathscr{S}$.

We claim that $P \subset S$. Otherwise, let $x \in P \backslash S$ and let S' be a minimal $\mathbf{X}$-subspace containing both x and S. Let $V \subset P \cap S$ be the S-interior of $P \cap S$. By definition V is not empty.

In the spherical case the antipodal point to x is not in $V \subset S$, since $x \notin S$. So for any point in V, there exists a unique shortest geodesic path joining it to x.

The whole "cone" based on V with vertex x is contained in $P \cap S'$ and this easily implies that x and $P \cap S$ are in the closure of the S'-interior of $P \cap S'$. This argument can be repeated for all $x \in (P \cap S') \backslash S$. Hence $S' \in \mathscr{S}$, which gives a contradiction.

Our claim is proved and the conclusion follows. $\square$

We define the *dimension* of an intersection P of half-spaces in $\mathbf{X}^n$ (in particular of a convex polyhedron) as the smallest integer i such that P is contained in an i-dimensional $\mathbf{X}$-subspace of $\mathbf{X}^n$. Lemma 2.3 shows that P is then thick in this subspace and the subspace is uniquely determined. A non-empty intersection of a convex polyhedron in $\mathbf{X}^n$ with an $\mathbf{X}$-subspace S of $\mathbf{X}^n$ is either a convex polyhedron in S or possibly a pair of antipodal points in the spherical case.

Let P be a convex polyhedron in $\mathbf{X}^n$. We define the *relative boundary* ∂P of P to be the topological boundary of P in S where S is the unique $\mathbf{X}$-subspace of $\mathbf{X}^n$ in which P is thick. The *relative interior* of P, denoted $\mathrm{RelInt}(P)$, is defined to be $P \setminus \partial P$. Both "interior" and "boundary" of P coincide with the topological interior and boundary respectively if and only if P is thick.

Let P be a convex polyhedron. A subset Q of ∂P is said to be a *codimension-one face* of P if P is thick in $\mathbf{X}^n$, $Q = P \cap S$ for some hyperplane S of $\mathbf{X}^n$, and Q is thick in S. (An exception has to be made when P is a semicircle and ∂P is a pair of antipodal points. In that case, we insist that Q is equal to one of the boundary points.) If $i \geqslant 2$, the *codimension-i* faces of P are defined inductively as codimension-one faces of codimension-$(i-1)$ faces of P. If P is thick in $\mathbf{X}^n$, a codimension-i face of P is a convex polyhedron of dimension $n-i$. Each codimension-i face of a convex polyhedron is contained in a face of codimension $i-1$.

LEMMA 2.4 (boundary). *Let P be a thick convex polyhedron in $\mathbf{X}^n$ which is the intersection of a locally finite family $\mathscr{H}$ of half-spaces. Then*

$$\partial P = \bigcup_{H \in \mathscr{H}} P \cap \partial H .$$

Proof of 2.4. Let $x \in \partial P$ and let U be an open neighbourhood of x. Let $\{H_1, \ldots, H_k\}$ be the set of elements of $\mathscr{H}$ whose boundary meets U. If U is small then k is finite, and we may assume that $x \in \partial H_i$ for $1 \leqslant i \leqslant k$. We must have $k \geqslant 1$, for, if $k = 0$, x would be in the interior of P in $\mathbf{X}^n$.

Conversely, if $x \in P \cap \partial H$ for some $H \in \mathscr{H}$, then x is in the topological boundary of P in $\mathbf{X}^n$. $\square$

PROPOSITION 2.5 (essential faces). *Let P and $\mathscr{H}$ be as in Lemma* 2.4. *Set*

$$\mathscr{M} = \left\{ H_0 \in \mathscr{H} : P \neq \bigcap_{H \in \mathscr{H} \setminus \{H_0\}} H \right\} .$$

Then:

(a) *P is the intersection of the elements of $\mathscr{M}$;*

(b) *the elements of* $\mathscr{M}$ *are characterized as the elements* H_0 *of* $\mathscr{H}$ *such that* $P \cap \partial H_0$ *is thick in* ∂H_0*;*

(c) *the set* $\mathscr{M}$ *of half-spaces depends only on* P *and not on* $\mathscr{H}$*.*

Note that neither Proposition 2.5 nor Lemma 2.4 need be true when the family of half-spaces is not locally finite. For example, the closed unit ball in $\mathbf{R}^n$ is the intersection of a countable family of half-spaces, none of whose boundaries meets the closed unit ball.

Proof of 2.5. Any element of $\mathscr{H} \setminus \mathscr{M}$ can be omitted from $\mathscr{H}$ without affecting P. It follows that any finite number of elements of $\mathscr{H} \setminus \mathscr{M}$ can be omitted without affecting P. Let P' be the intersection of the elements of $\mathscr{M}$. Then $P \subset P'$. If P' is not connected, then P' must consist of two antipodal points and P must be a single point. But this contradicts the definition of $\mathscr{M}$, and so P' is connected. By the local finiteness property, every point of P has a neighbourhood U such that $P \cap U = P' \cap U$. This shows that P is an open subset of P'. Since P' is connected and P is a non-empty closed subset of $\mathbf{X}^n$, $P = P'$.

Assume that $H_0 \in \mathscr{M}$. Let P_0 be the intersection of the elements of $\mathscr{H} \setminus \{H_0\}$ and choose $x \in P_0 \setminus P$. Consider an open set U internal to P, and let C be the cone over U with vertex x. As shown in Figure 1, $C \cap \partial H_0$ is contained in P and has non-empty interior in ∂H_0, which implies that $P \cap \partial H_0$ is thick in ∂H_0.

Conversely, if x is in the ∂H_0-interior of $P \cap \partial H_0$, the only half-space containing P and having x on its boundary is H_0. Therefore, if H_0 is omitted,

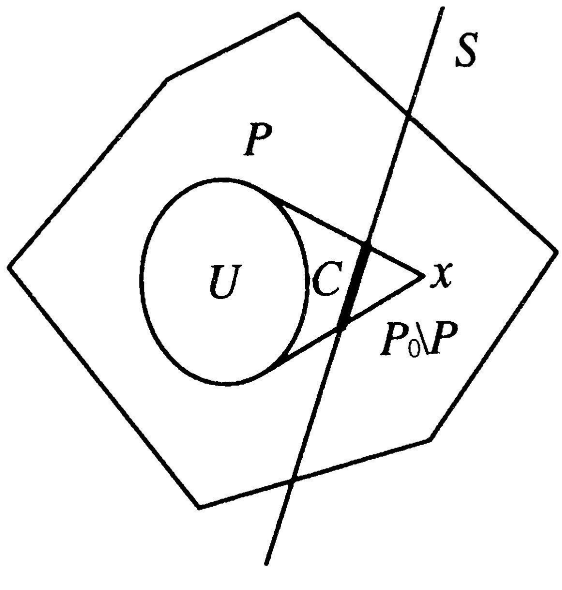

FIGURE 1.
Thick intersections.
If a half-space is essential for the definition of a polyhedron then its intersection with the polyhedron is thick. In the diagram the boundary ∂H_0 of H_0 is denoted by S.

x becomes an interior point of the intersection of half-spaces. So $H_0 \in \mathcal{M}$. The same argument proves that the elements of $\mathcal{M}$ can be characterized independently of $\mathcal{H}$ as the half-spaces H containing P and such that $P \cap \partial H$ is thick in ∂H. $\square$

The elements of the set $\mathcal{M}$ described in Proposition 2.5 are called the *essential half-spaces* of P. According to Proposition 2.5, the essential half-spaces are exactly those whose boundaries contain codimension-one faces of P. Lemma 2.4 implies the following result.

COROLLARY 2.6 (union of faces). *The boundary of a thick convex polyhedron in* $\mathbf{X}^n$ *is the union of its codimension-one faces.*

LEMMA 2.7 (codimension-two faces). *If* P *is a convex polyhedron in* $\mathbf{X}^n$ *and* C *is a codimension-two face of* P *there exist exactly two codimension-one faces of* P *containing* C.

Proof of 2.7. Without loss of generality we can assume P is thick in $\mathbf{X}^n$. Let S be the codimension-two subspace containing C. We may suppose that P is defined by its essential half-spaces. It follows from our definition of a face that there exist at least two essential half-spaces H_1 and H_2 whose boundary contains S. So C is contained in the codimension-one faces $P \cap \partial H_1$ and $P \cap \partial H_2$. Conversely if a codimension-one face $P \cap \partial H$ contains C then ∂H contains S. But it is easily checked (see Figure 2) that there cannot be three essential half-spaces whose boundaries have a codimension-two subspace in common. $\square$

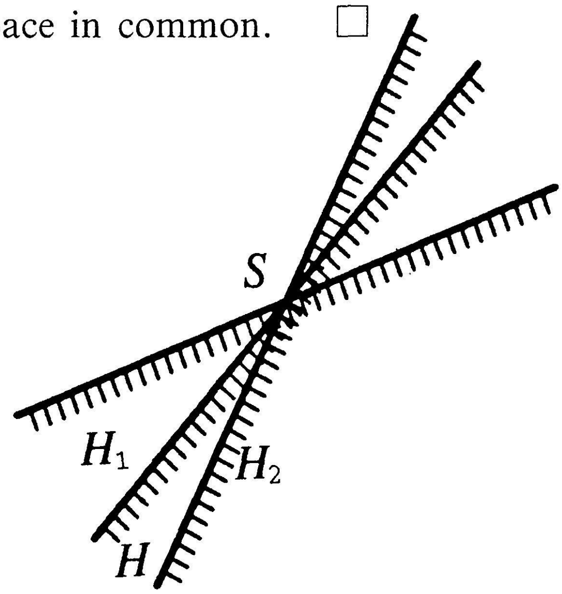

FIGURE 2.
Inessential half-spaces.
If three hyperplanes meet in a codimension-two subspace one of the corresponding half-spaces is not essential.

Let $n \geqslant 2$. A *dihedral region with corner* S is defined to be the intersection of two half-spaces, whose boundaries intersect in a subspace S of codimension two. The *dihedral angle* of the dihedral region is defined to be the angle between the boundaries. This is measured by taking a two-dimensional subspace orthogonal to S and seeing what angle is marked out on it by the boundaries. If we think of one half-space as first and the other as second, and if we orient the orthogonal plane, then the dihedral angle θ is signed and $0 < |\theta| < \pi$. The definition can be extended to the case where the boundaries of the half-spaces coincide. If the half-spaces themselves coincide, the angle is defined (ambiguously) to be $\pm\pi$, and if the half-spaces have the same boundary, but are otherwise disjoint, the angle is defined to be zero.

DEFINITION 2.8 (convex cell). A *convex cell* is a slight generalization of a convex polyhedron in $\mathbf{X}^n$; it is a convex polyhedron whose proper faces may have been subdivided. Formally, a convex cell is a convex polyhedron P in $\mathbf{X}^n$, together with a locally finite collection of convex polyhedra $\{P_i\}_{i \in I}$ satisfying the following conditions:

(a) The relative interiors of P and of the P_i, $(i \in I)$, form a disjoint covering of P.

(b) For each $i \in I$, P_i together with $\{P_j \mid j \in I,\ P_j \subset \partial P_i\}$ is a convex cell. (This definition is not circular since the dimension of P_i is smaller than that of P.)

The P_i are called the *faces* of the convex cell. By abuse of notation, we will often denote the convex cell by P, without mentioning the P_i. The most obvious example of a convex cell is a convex polyhedron, together with all its proper faces. A convex cell is said to be thick in $\mathbf{X}^n$ if the underlying polyhedron is thick in $\mathbf{X}^n$.

We now present some lemmas which will be useful in the sequel.

LEMMA 2.9 (positive distance 1). *Two disjoint affine subspaces of* $\mathbf{E}^n$ *have positive distance from each other.*

Proof of 2.9. Consider the orthogonal projection to an orthogonal complement of one of the subspaces, and note that distances are not increased. It follows that we can assume that one of the subspaces is a point, in which case the conclusion is obvious. □

LEMMA 2.10 (positive distance 2). *Let* S, T *be affine subspaces of* $\mathbf{E}^n$ *and let* $S \cap T = V \neq \varnothing$. *We assume that* $S \neq V$. *Let* $\varepsilon > 0$ *and define* $\hat{S}_\varepsilon = \{s \in S : d(s, V) \geqslant \varepsilon\}$. *Then* $d(\hat{S}_\varepsilon, T) > 0$.

Proof of 2.10. Assume first that the intersection V is a point. We may take $V = \{0\}$ with respect to the usual coordinates of $\mathbf{R}^n \cong \mathbf{E}^n$. As s varies in $S \backslash \{0\}$ and t varies in T, the distance between $s/\| s \|$ and t is bounded away from zero by compactness of the unit sphere in S. This proves the result when V is a point.

Now consider the general case. Let π be the projection on some orthogonal complement of V. Then

$$d(T, \hat{S}_\varepsilon) = d(\pi T, \pi \hat{S}_\varepsilon) > 0$$

as we see from the case where V is a point. □

PROPOSITION 2.11 (positive distance 3). *Let A and B be disjoint convex cells in the sphere or in euclidean space, each having only a finite number of faces. Then they are a positive distance apart.*

Proof of 2.11. This fact is obvious in the sphere, by compactness.

We prove the assertion by induction on the sum of the dimensions of A and B, which we denote by m. The case $m = 0$ is obvious, so we assume that $m > 0$ and that the assertion is true for all integers less than m. Assume by contradiction that there exist sequences $\{a_i\} \subset A$ and $\{b_i\} \subset B$ such that $d(a_i, b_i) \to 0$.

First of all we can assume that there is a $\delta > 0$, such that, for all i, $d(a_i \partial A) \geqslant \delta$; otherwise, using the fact that there are only finitely many faces, we can find a subsequence (which we denote by $\{a_i\}$ as well) and a proper face F of A such that $d(a_i, F) \to 0$; if we choose $\tilde{a}_i \in F$ such that $d(\tilde{a}_i, a_i) \to 0$, we have $d(\tilde{a}_i, b_i) \to 0$. The induction hypothesis applies to the faces F and B, proving that they meet, and this is a contradiction. Similarly, we can assume that the distance between the b_i's and ∂B is bounded away from 0; we can assume the same bound δ works for both.

Now, let S and T be the minimal subspaces containing A and B respectively. We claim that $S \not\subset T$ and $T \not\subset S$. Suppose for example that $S \subset T$, and choose i so that $d(a_i, b_i) < \delta$. Then $a_i \in T \cap B_\delta(b_i) \subset B$, which is false. So we assume that $S \neq T$. Lemma 2.9 implies that $V = S \cap T \neq \varnothing$, and Lemma 2.10 implies that $d(a_i, V) \to 0$. Then we can find $\{v_i\} \subset V$ such that $d(v_i, a_i) \to 0$, and hence $d(v_i, b_i) \to 0$. Since A is thick in S, as soon as $d(v_i, a_i) \leqslant \delta$ we have $v_i \in A$, and similarly if $d(v_i, b_i) \leqslant \delta$ we have $v_i \in B$. This is a contradiction. □

LEMMA 2.12 (constant multiple). *Let P be a convex polyhedron in $\mathbf{X}^n$ with only a finite number of faces. Let E be a face of P and* *let S be the subspace of $\mathbf{X}^n$ containing E in which E is thick. Then there exists a constant $k > 0$ such that, for $x \in P$, $d(x, S) \geqslant k \cdot d(x, E)$. (We make an exception of the case where S is a pair of antipodal points — the result may then be false.)*

Proof of 2.12. We will obtain a contradiction by assuming that there exists a sequence (x_i) in $P \backslash E$ for which $d(x_i, S)/d(x_i, E) \to 0$. In the spherical case $d(x_i, E) \leqslant \pi$, so $d(x_i, S) \to 0$. Using compactness, it follows that $d(x_i, E)$ also converges to zero. Therefore we may restrict our attention to an approximately euclidean local picture. So we assume from now on that we are in the hyperbolic or euclidean case.

Given $x \in P \backslash E$, let y be the nearest point in S and let z be the nearest point in E. Let E_0 be the face of E containing z in its relative interior. The geodesic xz is orthogonal to E_0 and xy is orthogonal to S. Moreover the segment yz meets E only at z.

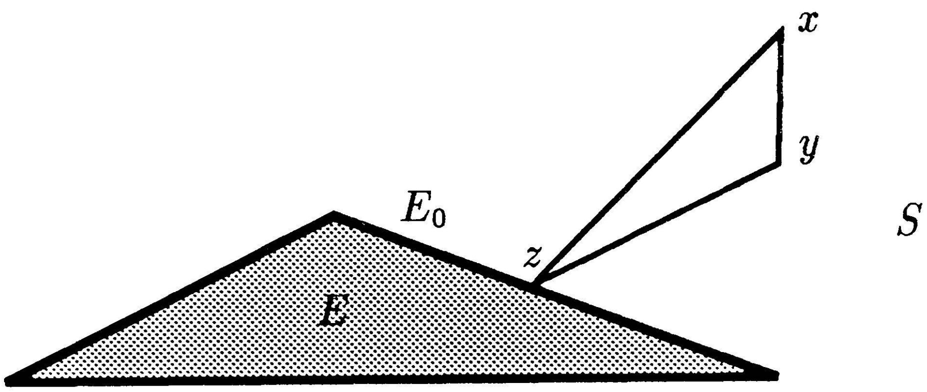

FIGURE 3.
Distance to face and subspace.
This picture illustrates the proof of Lemma 2.12. E is a face which is thick in the subspace S. The point z is the nearest point in E to x, and the smallest face containing z is E_0. The point y is the nearest point in S to x.

In our proof by contradiction, we obtain a sequence $x_i \in P \backslash E$ and corresponding sequences y_i and z_i, defined as above, such that $d(x_i, y_i)/d(x_i, z_i)$ converges to zero. This means that the angle between the segment $x_i z_i$ and S converges to zero. Since there are only a finite number of faces, we may assume that z_i lies in the relative interior of the same E_0 for each i. For each i, without changing the angle $\angle x_i z_i y_i$, we may now, without loss of generality, move x_i nearer to z_i along the ray $x_i z_i$, keeping z_i fixed and moving y_i correspondingly; z_i remains the nearest point of E. This moves y_i along the ray $y_i z_i$. The ratio $d(x_i, y_i)/d(x_i, z_i)$ is unaltered by the movement in the euclidean case, and is decreased in the hyperbolic case. We may therefore assume that, for each essential half-space H of P such that ∂H does not contain E_0 and for each i, $d(x_i, y_i) \leqslant d(x_i, z_i) < d(z_i, \partial H)/2$. Therefore $d(z_i, y_i) < d(z_i, \partial H)$.

It follows that for each essential half-space H of P, such that $y_i \notin H$, ∂H must contain E_0. Other half-spaces now become irrelevant, and we may assume that the boundary of each essential half-space of P contains E_0. The local geometry is therefore unchanged as z_i moves in $\mathrm{RelInt}(E_0)$, and we may assume without loss of generality that $z_i = z \in \mathrm{RelInt}(E_0)$ is independent of i.

Since only the angles are important, we may now move x_i along the ray zx_i, and assume that $d(x_i, z) = r$ is independent of i. Then z remains the nearest point of E to x_i and $d(x_i, y_i)$ tends to zero. We see that x_i converges (after taking a subsequence) to a point $x \in S$. It follows that $x \in P \cap S$ and therefore $x \in E$. But this contradicts the fact that z is the nearest point of E to x_i. This contradiction proves the result. □

LEMMA 2.13 (positive distance 4). *Let P, Q be convex polyhedra with a finite number of faces in $\mathbf{X}^n$ and let $E = P \cap Q$ be a non-empty face of both P and Q. We assume that $E \neq P$. Let $\delta > 0$ and denote by $N_\delta(E)$ the δ-neighbourhood of E. Then, provided δ is small enough so that P is not contained in $N_\delta(E)$, we have $d(P \backslash N_\delta(E), Q) > 0$.*

Proof of 2.13. The result is clearly true in the spherical case, so we assume that $\mathbf{X}^n$ is hyperbolic or euclidean space. Let S be the subspace containing E in which E is thick. By Lemma 2.12 there exists $\delta' > 0$ such that $P \backslash N_\delta(E) \subset P \backslash N_{\delta'}(S)$. Let P' be the intersection of the essential half-spaces of P whose boundary contains E and define Q' similarly. It is easy to check that $P' \cap Q' = S$ by considering a neighbourhood of a point of $\mathrm{RelInt}(E)$. Now $P \subset P'$ and $Q \subset Q'$, so it is sufficient to check that

$$d(P' \backslash N_{\delta'}(S), Q') > 0 \, .$$

But now, everything is invariant under isometries which preserve S and act trivially in the direction normal to S, so we can work in an orthogonal complement to S.

Hence we only have to check that the result holds when P is replaced by P', Q is replaced by Q' and $P' \cap Q' = E = S$ is a point. We argue by contradiction. Let $x_i \in P' \backslash N_{\delta'}(E)$ and $y_i \in Q'$, be sequences such that $d(x_i, y_i)$ converges to zero. We may clearly assume that $d(y_i, E) > \delta'/2$ for each i. The rays Ex_i and Ey_i (extended indefinitely) therefore converge to the same ray, which must lie in both P' and Q'. This contradicts the fact that $P' \cap Q' = E$. □

DEFINITION 2.14 (link). Let $(P, \{P_i\}_{i \in I})$ be a thick convex cell in $\mathbf{X}^n$, and let E be one of the faces of P — that is, E is equal to one of the $P_j, (j \in I)$. Let $J \subset I$ be the set of indices j such that P_j contains E. Let $p \in \mathrm{RelInt}(E)$. Let S_p be a sphere in $\mathbf{X}^n$ with centre p, whose radius is chosen small enough so that it only meets faces of P which contain E. By a change of scale, S_p can be identified with $\mathbf{S}^{n-1}$. The *link* of p in P is defined to be a convex cell in $\mathbf{S}^{n-1}$, given by $S_p \cap P$, with the face structure given by $S_p \cap P_j$. There is one exceptional situation we need to discuss, when E is one-dimensional. In that case, $S_p \cap E$ consists of two points, and this gives rise to two zero-dimensional faces in the link, not one. Note that if E is a point, then, for each $j \in J$, $P_j \cap S_p$ is a convex polyhedron in S_p — the exceptional case of two antipodal points cannot arise since E is in the relative boundary of P_j.

Notice that it does not matter where we choose $p \in \mathrm{RelInt}(E)$, as there is an isometry between the links given by two different choices. This means that up to isometry the link depends only on E and not on p.

3. CONDITIONS FOR POINCARÉ'S THEOREM

We describe in this section various conditions which come up when we are given a set of convex cells and instructions for glueing them together: our basic objective (see Remark 3.6) is to make orbifolds or manifolds from these building blocks. Alternatively, we can express our basic objective as constructing a tessellation of hyperbolic or euclidean space or the sphere.

Let $n \geqslant 2$. Let $\mathscr{P}$ be a countable or finite set of thick convex cells in $\mathbf{X}^n$.

REMARK 3.1.

(a) In fact we are only interested in the members of $\mathscr{P}$ up to isometry, and all our considerations must take this into account. This means that any $P \in \mathscr{P}$ may be replaced by $\psi(P)$, where $\psi \in \mathrm{Isom}(\mathbf{X}^n)$, and this must not affect any of our considerations in an essential way.

(b) Strictly speaking, the set $\mathscr{P}$ is an indexed set — that is, we allow repetition. One could avoid this, using Remark 3.1(a), by moving each repeated convex cell a little to a different place, but that seems artificial.

We denote by $\mathscr{F}(\mathscr{P})$ the set of all pairs (F, P) as P varies over $\mathscr{P}$ and F varies over the codimension-one faces of P. Notice that two faces of different convex cells could be geometrically coincident, but nonetheless they must be viewed as distinct according to Remark 3.1(a).