3. Conditions for Poincaré's Theorem

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DEFINITION 2.14 (link). Let $(P, \{P_i\}_{i \in I})$ be a thick convex cell in $\mathbf{X}^n$, and let E be one of the faces of P — that is, E is equal to one of the $P_j, (j \in I)$. Let $J \subset I$ be the set of indices j such that P_j contains E. Let $p \in \mathrm{RelInt}(E)$. Let S_p be a sphere in $\mathbf{X}^n$ with centre p, whose radius is chosen small enough so that it only meets faces of P which contain E. By a change of scale, S_p can be identified with $\mathbf{S}^{n-1}$. The *link* of p in P is defined to be a convex cell in $\mathbf{S}^{n-1}$, given by $S_p \cap P$, with the face structure given by $S_p \cap P_j$. There is one exceptional situation we need to discuss, when E is one-dimensional. In that case, $S_p \cap E$ consists of two points, and this gives rise to two zero-dimensional faces in the link, not one. Note that if E is a point, then, for each $j \in J$, $P_j \cap S_p$ is a convex polyhedron in S_p — the exceptional case of two antipodal points cannot arise since E is in the relative boundary of P_j.

Notice that it does not matter where we choose $p \in \mathrm{RelInt}(E)$, as there is an isometry between the links given by two different choices. This means that up to isometry the link depends only on E and not on p.

3. CONDITIONS FOR POINCARÉ'S THEOREM

We describe in this section various conditions which come up when we are given a set of convex cells and instructions for glueing them together: our basic objective (see Remark 3.6) is to make orbifolds or manifolds from these building blocks. Alternatively, we can express our basic objective as constructing a tessellation of hyperbolic or euclidean space or the sphere.

Let $n \geqslant 2$. Let $\mathscr{P}$ be a countable or finite set of thick convex cells in $\mathbf{X}^n$.

REMARK 3.1.

(a) In fact we are only interested in the members of $\mathscr{P}$ up to isometry, and all our considerations must take this into account. This means that any $P \in \mathscr{P}$ may be replaced by $\psi(P)$, where $\psi \in \mathrm{Isom}(\mathbf{X}^n)$, and this must not affect any of our considerations in an essential way.

(b) Strictly speaking, the set $\mathscr{P}$ is an indexed set — that is, we allow repetition. One could avoid this, using Remark 3.1(a), by moving each repeated convex cell a little to a different place, but that seems artificial.

We denote by $\mathscr{F}(\mathscr{P})$ the set of all pairs (F, P) as P varies over $\mathscr{P}$ and F varies over the codimension-one faces of P. Notice that two faces of different convex cells could be geometrically coincident, but nonetheless they must be viewed as distinct according to Remark 3.1(a).

CONDITION 3.2 (Pairing). Suppose we are given maps $R : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P})$ and $A : \mathcal{F}(\mathcal{P}) \to \mathrm{Isom}(\mathbf{X}^n)$ with the following properties:

(a) $R : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P})$ is an involution, that is $R \circ R$ is the identity.

(b) Let $(F, P) \in \mathcal{F}(\mathcal{P})$ and let $R(F, P) = (F', P')$. Then $A(F, P) \in \mathrm{Isom}(\mathbf{X}^n)$ maps F onto F' and maps the interior of P to the other side of F' from the interior of P'.

(c) $A(F, P)$ gives an isomorphism between the face structure of F and the face structure of F'.

(d) For each $(F, P) \in \mathcal{F}(\mathcal{P})$, $A(R(F, P)) = A(F, P)^{-1}$.

In that case, we say that (R, A) is a *face-pairing* for $\mathcal{P}$, and say the condition Pairing$(\mathcal{P}, R, A)$ is satisfied. (R, A) is also known as *glueing data.*

REMARK 3.3 (order two). In case $R(F, P) = (F, P)$ Condition 3.2(d) implies that $A(F, P)$ is a mapping of order two. Note that in this special situation $A(F, P)$ is not necessarily the reflection in the face F, though that is a common application of this theory.

EXAMPLE 3.4 (triangle example). Consider an equilateral triangle P in $\mathbf{E}^2$, and let $\mathcal{P} = \{P\}$. In this case a face-pairing is an isometry sending an edge to itself or another edge. For each pair of edges there are four such isometries of $\mathbf{E}^2$, but two of the four are excluded by Condition 3.2(b). This enables one to easily list all possible sets of face-pairings. (In fact there are twenty distinct sets of face-pairings.)

CONDITION 3.5 (connected). Connected$(\mathcal{P}, R)$ is the condition that, given any two convex cells P and P' in $\mathcal{P}$, there exists a finite sequence of elements $\{(F_i, F_i', P_i)\}_{i=1,\dots,k}$ with $P_i \in \mathcal{P}$ and F_i and F_i' codimension-one faces of P_i, such that $P_1 = P$, $P_k = P'$ and $R(F_i', P_i) = (F_{i+1}, P_{i+1})$ for $i \geqslant 1$. This condition means that any two elements of $\mathcal{P}$ are joined by a sequence of face-pairings.

REMARK 3.6 (basic objective). If Pairing$(\mathcal{P}, R, A)$, we can glue up $\mathcal{P}$ and obtain an identification space $Q = Q(\mathcal{P}, R, A)$. If we remove the $(n-2)$-skeleton, we obtain a manifold M modelled on $\mathbf{X}^n$ which falls into pieces if we remove the $(n-1)$-skeleton; each piece is the interior of some $P \in \mathcal{P}$. The universal cover of M is also divided into cells, each of which is isometric to (the interior of) some element $P \in \mathcal{P}$. If Connected$(\mathcal{P}, R)$, then M is connected, and its universal cover is mapped into $\mathbf{X}^n$ by the developing map. Different cells in the universal cover will in general correspond to the same P, because M is not simply connected. The developing map is uniquely defined, once the map is fixed on one component of the inverse image of one element of $\mathscr{P}$. Roughly speaking, our basic objective is to find conditions such that the closures in $\mathbf{X}^n$ of the images of the cells of the universal cover tessellate $\mathbf{X}^n$.

More precisely, we start with countably many copies of the elements of $\mathscr{P}$ and lay them out in $\mathbf{X}^n$ one by one. Each new copy has to be glued to a free face of what is already laid out, using the appropriate (conjugate of the) face-pairing. If at any stage overlapping of interiors occurs, or if the boundaries intersect, but not in a common face, or if a face of the new copy coincides with some existing free face, but not according to one of the given face-pairings, then the process fails. The process succeeds if we end with a locally finite tessellation of the whole of $\mathbf{X}^n$. The process might continue indefinitely without failure at any finite stage, for example covering a proper subspace of $\mathbf{X}^n$, and it will have failed if at the end it does not give a locally finite tessellation of all of $\mathbf{X}^n$.

We now describe some more conditions which arise in considering Poincaré's Theorem. Suppose Pairing$(\mathscr{P}, R, A)$. Let $(F_1, P_1) \in \mathscr{F}(\mathscr{P})$ and let C_1 be a codimension-one face of F_1. Let F_1' be the other codimension-one face of P_1 containing C_1 (see Lemma 2.7). Let $R(F_1', P_1) = (F_2, P_2)$ and let $g_1 = A(F_1', P_1)$. Note that $C_2 = g_1(C_1)$ is a codimension-one face of F_2, so it is a codimension-two face of P_2, and hence there exists only one other codimension-one face of P_2 containing C_2. We call this face F_2'. Set $g_2 = A(F_2', P_2)$ and continue in the same way, obtaining a sequence $\{\sigma_i = (P_i, C_i, F_i, F_i', g_i)\}_{i=1,2,\ldots}$. We have $g_i = A(F_i', P_i)$ and $g_{i-1} \circ \ldots \circ g_1(C_1) = C_i$. The sequence is determined once one has chosen P_1, F_1 and C_1.

Condition 3.7 (FirstCyclic). FirstCyclic$(\mathscr{P}, R, A)$ is the condition that, for each $(F_1, P_1) \in \mathscr{F}(\mathscr{P})$ and for each codimension-one face C_1 of F_1, there is some $r \geqslant 1$ such that $\sigma_{r+1} = \sigma_1$. The minimal $r \geqslant 1$ with this property is called the *first cycle length* of (C_1, F_1, P_1).

Remark 3.8.

(a) The condition $\sigma_{r+1} = \sigma_1$ is obviously equivalent to the conditions $P_{r+1} = P_1$, $C_{r+1} = C_1$ and $F_{r+1} = F_1$.

(b) Instead of starting with P_1, F_1 and C_1, we could instead start with P_i, F_i and C_i, or with P_i, F_i' and C_i. Instead of getting the r-tuple

$(\sigma_1, \ldots, \sigma_r)$, we would get a cyclic permutation of it, or a cyclic permutation of $(\sigma'_r, \ldots, \sigma'_1)$, where $\sigma'_i = (P_i, C_i, F'_i, F_i, g_{i-1}^{-1})$ for $1 \leqslant i \leqslant r$ and the indices are interpreted mod r.

(c) FirstCyclic($\mathscr{P}$, R, A) clearly has to be satisfied if our basic objective is to be achieved (see Remark 3.6). Note however that complications arise if we do not insist on local finiteness in the definition of a tessellation, when formulating our basic objective. For example, in $\mathbf{E}^2$, we could glue together a countable number of wedges, such that the sum of the wedge angles is 2π. Such a construction would not give the whole of $\mathbf{E}^2$, but would leave a single ray uncovered: is this a tessellation? The meaning of the word "tessellation" does not suffer from such ambiguities when one insists on local finiteness of the face structure.

CONDITION 3.9 (finite). Finite($\mathscr{P}$) is the condition that $\mathscr{P}$ is finite and that each element of $\mathscr{P}$ has only a finite number of faces. This is one of the usual conditions imposed for Poincaré's Theorem, but it is clearly not essential for our basic objective (see Remark 3.6). However, this condition is essential if one wishes to check all the conditions by a finite mechanical procedure.

Clearly, if Finite($\mathscr{P}$) then FirstCyclic($\mathscr{P}$, R, A).

CONDITION 3.10 (SecondCyclic). SecondCyclic($\mathscr{P}$, R, A) is the condition that for each $(F_1, P_1) \in \mathscr{F}(\mathscr{P})$ and for each codimension-one face C_1 of F_1, there exists $r \geqslant 1$ such that $\sigma_{r+1} = \sigma_1$ and the restriction of $g_r \circ \ldots \circ g_1$ to C_1 is the identity. The minimal $r \geqslant 1$ with this property is called the *second cycle length* of (C_1, F_1, P_1). Even if FirstCyclic($\mathscr{P}$, R, A), the second cycle length may be infinite (see Example 3.32 or Example 3.17).

The reader is referred to Examples 3.15, 3.16, 3.17 and 3.18, which may provide a better understanding of the cycle conditions.

REMARK 3.11.

(a) According to Remark 3.1(a), we need to check that our condition is not changed by the replacement of one of the convex cells $P \in \mathscr{P}$ by $\psi(P)$ for some $\psi \in \mathrm{Isom}(\mathbf{X}^n)$. In fact, suppose $(F', P') \in \mathscr{F}(\mathscr{P})$ and $R(F', P') = (F'', P'')$. Then $A(F', P')$ must be replaced by:

- $\psi \circ A(F', P') \circ \psi^{-1}$ if $P' = P$ and $P'' = P$;
- $A(F', P') \circ \psi^{-1}$ if $P' = P$ and $P'' \neq P$;
- $\psi \circ A(F', P')$ if $P' \neq P$ and $P'' = P$;
- $A(F', P')$ if $P' \neq P$ and $P'' \neq P$.

It follows that the mapping $g = g_r \circ \dots \circ g_1$ to which SecondCyclic($\mathscr{P}, R, A$) refers is either unchanged or is replaced by $\psi \circ g \circ \psi^{-1}$ under the replacement of P by $\psi(P)$, so the condition is well-defined.

(b) Just as in the case of first cycles, the second cycle length will be the same for each (F_i, P_i) and (F_i', P_i) which occurs in the cycle. The mapping g of Remark 3.11(a) has to be replaced by g^{-1} when (F_1, P_1) is replaced by (F_1', P_1). As we have seen in Remark 3.11(a), g is only defined in an intrinsic way up to conjugation, because each of the convex cells is only defined up to isometry. If we start with (F_i, P_i) instead of (F_1, P_1), then g once again changes by a conjugation.

(c) FirstCyclic($\mathscr{P}, R, A$) and SecondCyclic($\mathscr{P}, R, A$) clearly have to be satisfied if our basic objective is to be achieved (see Remark 3.6).

LEMMA 3.12 (cycles and rotations). *We use the notation introduced above and assume that* $r \geqslant 1$ *is the second cycle length of* (C_1, F_1, P_1). *Let* θ_i *be the the dihedral angle of* P_i *along* C_i *for* $i = 1, \dots, r$. *Then the isometry* $g = g_r \circ \dots \circ g_1$ *of* $\mathbf{X}^n$ *is a rotation through an angle* $\Sigma\theta_i$ *around the codimension-two subspace of* $\mathbf{X}^n$ *containing* C_1.

Proof of 3.12. We denote by S the codimension-two subspace containing C_1. Note that g is necessarily the identity on S, since C_1 has non-empty S-interior and $g \mid C_1$ is the identity by hypothesis. We only need to prove that g preserves the orientation.

Consider in $\mathbf{X}^n$ the convex cells $P_1, g_1^{-1}(P_2), \dots, (g_1^{-1} \circ \dots \circ g_r^{-1})(P_{r+1})$; they have a common codimension-two face

$$C_1 = g_1^{-1}(C_2) = \dots = (g_1^{-1} \circ \dots \circ g_r^{-1})(C_{r+1}) .$$

Moreover, according to Condition 3.2(b), $(g_1^{-1} \circ \dots \circ g_{i-1}^{-1})(P_i)$ and $(g_1^{-1} \circ \dots \circ g_i^{-1})(P_{i+1})$ lie on opposite sides of the common codimension-one face $(g_1^{-1} \circ \dots \circ g_{i-1}^{-1})(F_i') = (g_1^{-1} \circ \dots \circ g_i^{-1})(F_{i+1})$. Fix an orientation for two-dimensional subspaces normal to S. If we assume that the angle from F_1 to F_1' is positive, then the angle from $F_1' = g_1^{-1}(F_2)$ to $g_1^{-1}(F_2')$ is also positive. By induction the angle from $(g_1^{-1} \circ \dots \circ g_r^{-1})(F_{r+1})$ to $(g_1^{-1} \circ \dots \circ g_r^{-1})(F_{r+1}')$ is positive. But $F_{r+1} = F_1$ and $F_{r+1}' = F_1'$, and hence $g_1^{-1} \circ \dots \circ g_r^{-1}$ preserves the orientation, as required. □

CONDITION 3.13 (ThirdCyclic). ThirdCyclic($\mathscr{P}, R, A$) is the condition that for all $(F_1, P_1) \in \mathscr{F}(\mathscr{P})$ and for all codimension-one faces C_1 of F_1, if $r \geqslant 1$ is the second cycle length, the mapping g described in Lemma 3.12 is a rotation through an angle of the form $2\pi/m$ for some non-zero $m \in \mathbf{Z}$.

REMARK 3.14. It follows from Remark 3.11(a) that this condition and the absolute value of m are both independent of $i (1 \leqslant i \leqslant r)$ and of whether one starts with (F_i, P_i) or (F'_i, P_i). The condition is necessary if our basic objective (see Remark 3.6) is to be achieved. However, we have to proceed carefully, as the following example shows. We take a wedge in $\mathbf{E}^2$, with angle $2\pi/3$. If the face-pairings are reflections, then the sum of angles which occurs in Condition 3.13 is $4\pi/3$, which is not of the required form. Note that the images of the wedge do tile $\mathbf{E}^2$. However, this tessellation is not consistent with the face-pairings (see Figure 4).

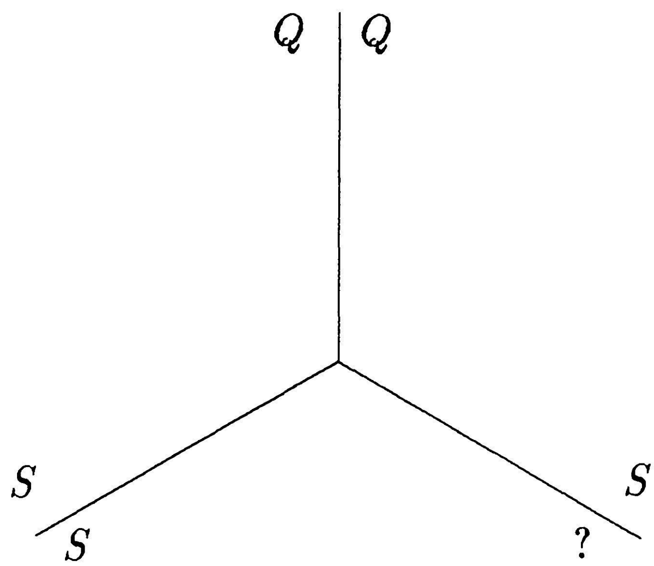

FIGURE 4.
Reflection in the sides of a wedge.
The different images seem to tessellate.
But if we take the face-pairings into account we find an inconsistency.

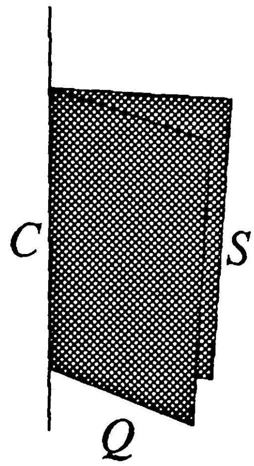

FIGURE 5.
Dihedral region.
This shows a dihedral region in $\mathbf{E}^3$, which is the only member of $\mathscr{P}$ in Examples 3.15, 3.16, 3.17 and 3.18.

EXAMPLE 3.15 (cyclic example 1). In $\mathbf{E}^3$ let P be the dihedral region with angle φ shown in Figure 5, and let $\mathscr{P} = \{P\}$. Let the codimension-one faces of P be Q and S, intersecting in the codimension-two face C. We set $R(Q, P) = (Q, P)$ and $R(S, P) = (S, P)$ and we define $A(Q, P)$ (respectively $A(S, P)$) to be the reflection in the plane containing Q (respectively S). Pairing($\mathscr{P}, R, A$) follows. Moreover, as illustrated in Figure 6, both first and second cycle lengths are equal to two. Then (by Lemma 3.12), ThirdCyclic($\mathscr{P}, R, A$) is equivalent to the condition $\varphi = \pi/m$ for some non-zero $m \in \mathbf{Z}$.

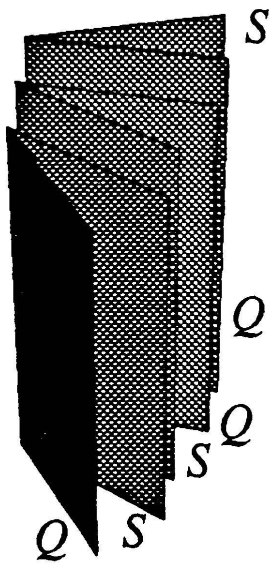

FIGURE 6.
Reflection face-pairings.
This illustrates Example 3.15. The first two cyclic conditions hold with $r = 2$.

EXAMPLE 3.16 (cyclic example 2). Let $\mathscr{P}$ be as in Example 3.15, set $R(Q, P) = (S, P)$ and define $A(Q, P)$ as the rotation through an angle φ around C; Pairing($\mathscr{P}, R, A$) is of course satisfied and FirstCyclic($\mathscr{P}, R, A$), SecondCyclic($\mathscr{P}, R, A$) both hold with $r = 1$ (see Figure 7). Hence, using

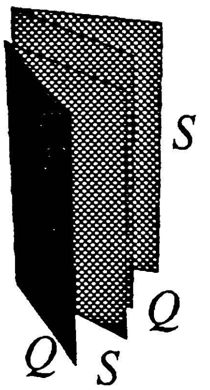

FIGURE 7.
Rotation face-pairing.
This illustrates Example 3.16. The first two cyclic conditions hold with $r = 1$.

Lemma 3.12, we see that ThirdCyclic($\mathscr{P}, R, A$) is equivalent to the condition that $\varphi = 2\pi/m$ for some non-zero $m \in \mathbf{Z}$.

EXAMPLE 3.17 (cyclic example 3). Let $\mathscr{P}$ and R be as in Example 3.16 and define $A(Q, P)$ as the composition of the rotation through an angle φ around C with a non-zero translation parallel to C. Then Pairing($\mathscr{P}, R, A$) is satisfied, FirstCyclic($\mathscr{P}, R, A$) is satisfied with $r = 1$ but SecondCyclic($\mathscr{P}, R, A$) is not satisfied.

EXAMPLE 3.18 (cyclic example 4). Let $\mathscr{P}$ and R be as in Example 3.16 and define $A(Q, P)$ as the composition of the rotation through an angle φ around C with the reflection in a plane orthogonal to C; Pairing($\mathscr{P}, R, A$) is satisfied. As shown in Figure 8, FirstCyclic($\mathscr{P}, R, A$) is satisfied with $r = 1$ (and hence for all $r \geqslant 1$), while SecondCyclic($\mathscr{P}, R, A$) is satisfied with $r = 2$. As in Example 3.15, ThirdCyclic($\mathscr{P}, R, A$) is equivalent to the condition that $\varphi = \pi/m$ for some non-zero $m \in \mathbf{Z}$.

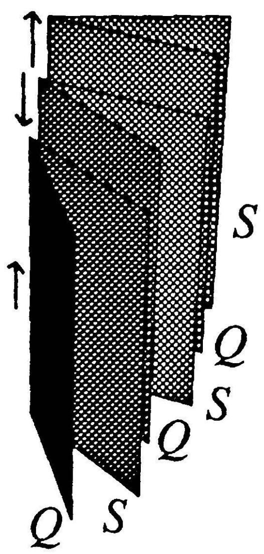

FIGURE 8.
Rotation plus flip face-pairing.
This illustrates Example 3.18.
The first cyclic condition holds with $r = 1$ and the second one with $r = 2$.

CONDITION 3.19 (Cyclic). Cyclic($\mathscr{P}, R, A$) is the conjunction of FirstCyclic($\mathscr{P}, R, A$), SecondCyclic($\mathscr{P}, R, A$) and ThirdCyclic($\mathscr{P}, R, A$).

We now introduce two more conditions, each of which involves the metric structure of the elements of $\mathscr{P}$.

CONDITION 3.20 (FirstMetric). FirstMetric($\mathscr{P}$) is the condition that there should exist a number $\varepsilon > 0$ such that for all elements P of $\mathscr{P}$ and for all faces E_1, E_2 of P, if $E_1 \cap E_2 = \varnothing$ then $d(E_1, E_2) \geqslant \varepsilon$ (where d denotes the usual distance between subsets of a metric space).

EXAMPLE 3.21 (not FirstMetric). FirstMetric($\mathscr{P}$) is not necessary for our basic objective to be achieved (see Remark 3.6). For example, take any tessellation of the euclidean plane by triangles. We can insert small triangles around the vertices, making the size of the inserted triangles tend to zero as one goes to infinity, as in Figure 9.

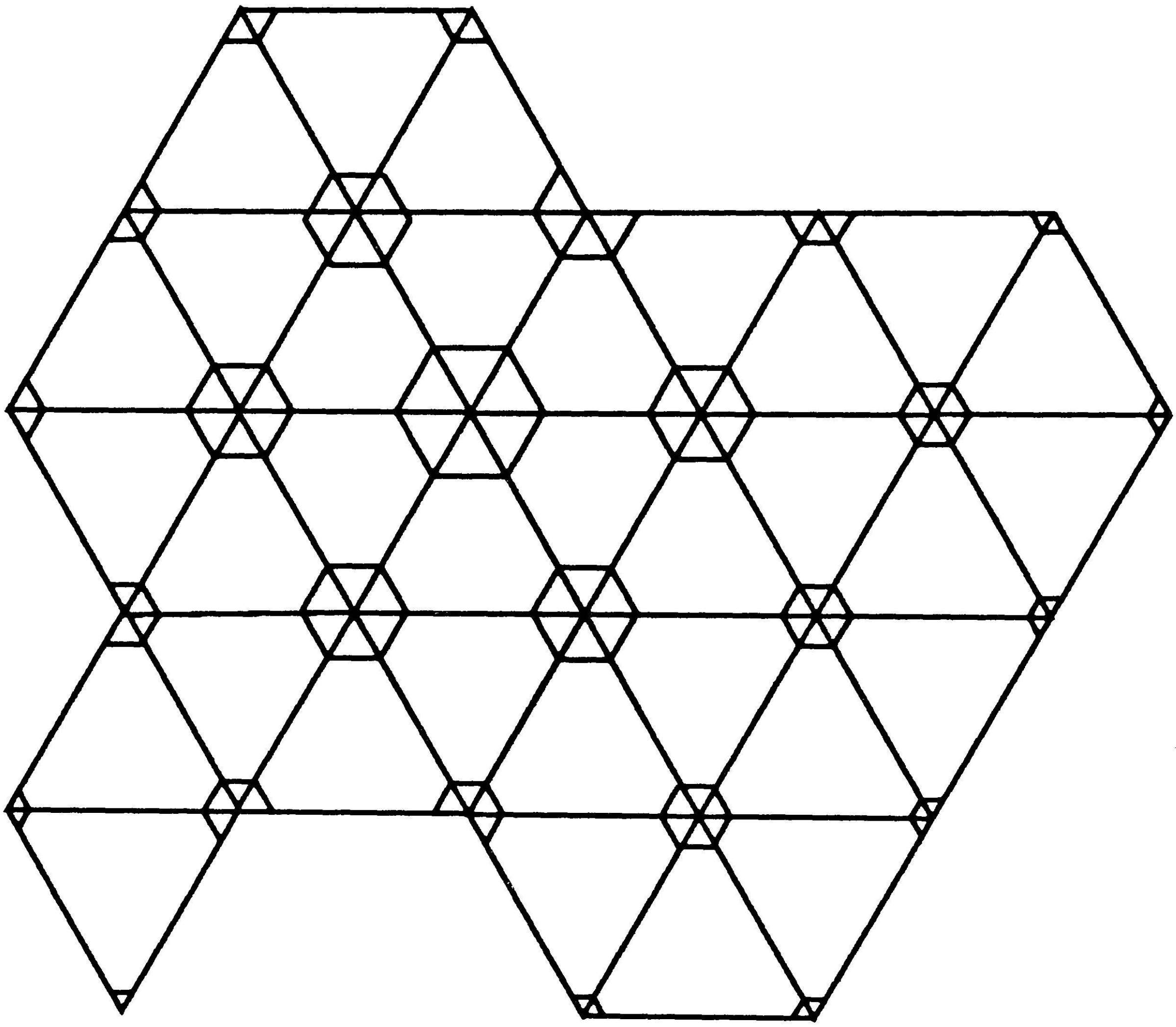

FIGURE 9.
Tessellation of $\mathbf{E}^2$. This illustrates Example 3.21.

CONDITION 3.22 (SecondMetric). SecondMetric($\mathscr{P}$) is the condition that, given any $\delta > 0$, there should exist $\mu(\delta) > 0$ with the following property. Suppose $P \in \mathscr{P}$ and E and F are faces of P such that $E \cap F \neq \varnothing$ and $E \not\subset F$. If x is a point of E at distance at least δ from ∂E, then $d(x, F) \geqslant \mu(\delta)$.

CONDITION 3.23 (Metric). Metric($\mathscr{P}$) is the conjunction of FirstMetric($\mathscr{P}$) and SecondMetric($\mathscr{P}$).

Condition 3.22 (SecondMetric) may appear to be strictly stronger than Condition 3.20 (FirstMetric), but it is not. For example, in $\mathbf{H}^3$, take P to be the intersection of four half-spaces, whose boundaries meet at a point z at infinity. We arrange for the intersection of P with a horosphere centred at p to be a square. Then SecondMetric($\mathscr{P}$) is satisfied but not FirstMetric($\mathscr{P}$).

REMARK 3.24 (Finite implies Metric). If Finite($\mathscr{P}$) holds then FirstMetric($\mathscr{P}$) is equivalent to the condition that any pair of disjoint faces of the elements of $\mathscr{P}$ have positive distance from each other. This is true in euclidean and spherical geometry but not necessarily true in hyperbolic geometry. For example, in the hyperbolic plane we take $\mathscr{P} = \{P\}$, where P is the region between two disjoint geodesics. If the geodesics meet at infinity then FirstMetric($\mathscr{P}$) is false. From Proposition 2.11 we see that Finite($\mathscr{P}$) implies FirstMetric($\mathscr{P}$), unless $\mathbf{X}^n = \mathbf{H}^n$. From Lemma 2.13 we see that Finite($\mathscr{P}$) implies SecondMetric($\mathscr{P}$) for all three geometries. Hence Finite($\mathscr{P}$) implies Metric($\mathscr{P}$) unless $\mathbf{X}^n = \mathbf{H}^n$.

SecondMetric($\mathscr{P}$) should be thought of as showing that the angle between faces is bounded below.

EXAMPLE 3.25. To make an example where FirstMetric($\mathscr{P}$) is satisfied, but not SecondMetric($\mathscr{P}$), we take a sequence of disjoint isoceles triangles T_i in $\mathbf{E}^2$, tending to infinity. T_i is chosen so that the apex angle tends to zero and the base of T_i always has length one, which means that the two equal sides have length tending to infinity. We can then complete this to a triangulation of $\mathbf{E}^2$ in which FirstMetric($\mathscr{P}$) is satisfied. SecondMetric($\mathscr{P}$) clearly fails.

Given a set S in $\mathbf{H}^n$ we denote by $\bar{S}$ the closure of S as a subset of $\bar{\mathbf{H}}^n$, and we refer to the points of $\bar{S} \cap \partial\mathbf{H}^n$ as the *points at infinity of S.*

LEMMA 3.26. *Let* P *be a convex cell in* $\mathbf{H}^n$ *with finitely many faces. Two disjoint faces of* P *can have at most one common point at infinity, and they are a positive distance apart if and only if they have no common point at infinity.*

Proof of 3.26. Let A and B be the two disjoint faces. If they have two common points at infinity, the geodesic joining them lies in both faces, contradicting the hypothesis that they are disjoint in $\mathbf{H}^n$. If A and B have a common point at infinity, then they are clearly zero distance apart. If, conversely, they are zero distance apart, then there are sequences $\{a_i\}$ in A and $\{b_i\}$ in B, such that $d(a_i, b_i)$ converges to zero. We may assume that the two sequences converge to the same point p at infinity. Then $p \in \bar{A} \cap \bar{B}$ as required. □

EXAMPLE 3.27. Consider the polygon $P \subset \mathbf{H}^2$ given in the upper half-plane model by $[1, 2] \times (0, \infty)$. The two faces of P have a common point at infinity, and they are zero distance apart. Multiplication by two induces a face-pairing which satisfies Pairing($\mathscr{P}, R, A$) and Cyclic($\mathscr{P}, R, A$). However the images of P under the powers of the pairing cover only the right half of the half-plane.

DEFINITION 3.28 (codimension-i graph). For $i \geqslant 1$ we define $\mathscr{F}^i(\mathscr{P})$ to be the set of all pairs (E, P) where $P \in \mathscr{P}$ and E is a codimension-i face of P. So $\mathscr{F}^1(\mathscr{P}) = \mathscr{F}(\mathscr{P})$. Given a face-pairing (R, A) we define a graph $\Gamma^i(\mathscr{P}, R, A)$ which has a vertex for each element (E, P) of $\mathscr{F}^i(\mathscr{P})$ and an edge $e(E, F, P)$ for each triple with $E \subset F \subset P$, E a codimension-i face of P and F a codimension-one face of P. The edge $e(E, F, P)$ joins (E, P) to (E', P') if $R(F, P) = (F', P')$ and $E' = A(F, P)(E)$; we regard $e(E, F, P)$ as being the same edge as $e(E', F', P')$. Each component of $\Gamma^1(\mathscr{P}, R, A)$ consists of one or two vertices and one edge. FirstCyclic($\mathscr{P}, R, A$) is equivalent to the condition that each component of $\Gamma^2(\mathscr{P}, R, A)$ is finite.

CONDITION 3.29 (LocallyFinite). We now describe a condition which is clearly necessary for our basic objective (see Remark 3.6). In many situations, this condition does not need to be explicitly verified, since it follows from various subsets of the other conditions. LocallyFinite($\mathscr{P}, R, A$) is the condition that each component of $\Gamma^i(\mathscr{P}, R, A)$ is a finite graph. Clearly Finite($\mathscr{P}$) implies LocallyFinite($\mathscr{P}, R, A$).

If $n = 2$, LocallyFinite($\mathscr{P}, R, A$) is equivalent to FirstCyclic($\mathscr{P}, R, A$).

EXAMPLE 3.30 (not LocallyFinite). Pairing($\mathscr{P}, R, A$), Connected($\mathscr{P}, R$) and Cyclic($\mathscr{P}, R, A$) do not imply LocallyFinite($\mathscr{P}, R, A$). An example may be constructed as follows. For each integer $n > 0$, take the two-sphere of radius $1/n$ in $\mathbf{R}^3$ lying above the plane $z = 0$ and tangent to it at 0. These spheres cut $\mathbf{R}^3$ into a countable number of pieces. We can approximate each piece by a finite union of convex polyhedra, so that everything fits together in the same qualitative fashion as the spheres we have described. (We first approximate the spherical surfaces, and then cut up the regions.) In particular the origin appears as a point in each of the approximations. The result is not locally finite at the origin, though the other hypotheses are satisfied. Note that, with the obvious path metric induced by gluing the pieces together, the resulting space is a complete metric space; so completeness does not help, in this type of situation, in deducing local finiteness

REMARK 3.31 (stronger local finiteness). There is an alternative version of the local finiteness condition, used for example in [Mas71]: recall from Remark 3.6 that $Q(\mathscr{P}, R, A)$ is the quotient space of $\bigsqcup_{P \in \mathscr{P}} P$, the disjoint union of the convex cells in $\mathscr{P}$. We might assume that the inverse image under the quotient map of any point in $Q(\mathscr{P}, R, A)$ is finite. This obviously implies LocallyFinite$(\mathscr{P}, R, A)$. It will turn out that LocallyFinite$(\mathscr{P}, R, A)$ together with Cyclic$(\mathscr{P}, R, A)$ implies this stronger condition (see Theorem 4.14).

EXAMPLE 3.32 (irrational). Here is an example when the weaker condition of local finiteness is true, but not the stronger condition. Of course, Cyclic$(\mathscr{P}, R, A)$ is not true in this case. We take two codimension-one spherical subspaces of $\mathbf{S}^3$. These meet along a common $\mathbf{S}^1$. Let P be one of the four complementary three-dimensional regions, and let $\mathscr{P} = \{P\}$. Then P has two faces, each of which is a hemisphere. Suppose we glue one of these hemispheres to the other, inducing an irrational rotation on the common circle boundary. Then we have LocallyFinite$(\mathscr{P}, R, A)$ and Finite$(\mathscr{P})$, but the strong version of local finiteness just stated is false.

Another similar example in $\mathbf{H}^4$ is given as follows. Take the intersection of two half-spaces, such that the boundaries of these half-spaces intersect in a hyperbolic plane. There are two codimension-one faces F_1 and F_2, each of which is half of a three-dimensional hyperbolic space, and one codimension-two subspace S, which is a hyperbolic plane. We take as a face-pairing a rotation keeping the codimension-two face S pointwise fixed and taking F_1 to F_2, followed by an isometry T of $\mathbf{H}^4$. T sends S to itself and is elliptic, rotating S through an irrational angle. If we take $\mathbf{H}^4$ to be embedded as one sheet of the hyperboloid $\langle v, v \rangle = -1$ in a five-dimensional vector space with indefinite inner product of type (4, 1), then T is the identity on $S^\perp$. Cyclic$(\mathscr{P}, R, A)$ is false, LocallyFinite$(\mathscr{P}, R, A)$ and Finite$(\mathscr{P})$ are true, but the quotient space Q is not hausdorff.

4. DEVELOPING MAPS

As in the previous section, let $\mathscr{P}$ be a set of thick convex cells in $\mathbf{X}^n$, and let (R, A) satisfy Pairing$(\mathscr{P}, R, A)$. We define a graph $\Gamma(\mathscr{P}, R)$ in the following way. The vertices of the graph are the elements of $\mathscr{P}$. We have an edge, which we call either $e(F, P)$ or $e(F', P')$, joining P and P' if and only if $R(F, P) = (F', P')$. So there is one edge for each face-pairing. Clearly, Connected$(\mathscr{P}, R)$ if and only if $\Gamma(\mathscr{P}, R)$ is connected.