4. Developing maps

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REMARK 3.31 (stronger local finiteness). There is an alternative version of the local finiteness condition, used for example in [Mas71]: recall from Remark 3.6 that $Q(\mathscr{P}, R, A)$ is the quotient space of $\bigsqcup_{P \in \mathscr{P}} P$, the disjoint union of the convex cells in $\mathscr{P}$. We might assume that the inverse image under the quotient map of any point in $Q(\mathscr{P}, R, A)$ is finite. This obviously implies LocallyFinite$(\mathscr{P}, R, A)$. It will turn out that LocallyFinite$(\mathscr{P}, R, A)$ together with Cyclic$(\mathscr{P}, R, A)$ implies this stronger condition (see Theorem 4.14).

EXAMPLE 3.32 (irrational). Here is an example when the weaker condition of local finiteness is true, but not the stronger condition. Of course, Cyclic$(\mathscr{P}, R, A)$ is not true in this case. We take two codimension-one spherical subspaces of $\mathbf{S}^3$. These meet along a common $\mathbf{S}^1$. Let P be one of the four complementary three-dimensional regions, and let $\mathscr{P} = \{P\}$. Then P has two faces, each of which is a hemisphere. Suppose we glue one of these hemispheres to the other, inducing an irrational rotation on the common circle boundary. Then we have LocallyFinite$(\mathscr{P}, R, A)$ and Finite$(\mathscr{P})$, but the strong version of local finiteness just stated is false.

Another similar example in $\mathbf{H}^4$ is given as follows. Take the intersection of two half-spaces, such that the boundaries of these half-spaces intersect in a hyperbolic plane. There are two codimension-one faces F_1 and F_2, each of which is half of a three-dimensional hyperbolic space, and one codimension-two subspace S, which is a hyperbolic plane. We take as a face-pairing a rotation keeping the codimension-two face S pointwise fixed and taking F_1 to F_2, followed by an isometry T of $\mathbf{H}^4$. T sends S to itself and is elliptic, rotating S through an irrational angle. If we take $\mathbf{H}^4$ to be embedded as one sheet of the hyperboloid $\langle v, v \rangle = -1$ in a five-dimensional vector space with indefinite inner product of type (4, 1), then T is the identity on $S^{\perp}$. Cyclic$(\mathscr{P}, R, A)$ is false, LocallyFinite$(\mathscr{P}, R, A)$ and Finite$(\mathscr{P})$ are true, but the quotient space Q is not hausdorff.

4. DEVELOPING MAPS

As in the previous section, let $\mathscr{P}$ be a set of thick convex cells in $\mathbf{X}^n$, and let (R, A) satisfy Pairing$(\mathscr{P}, R, A)$. We define a graph $\Gamma(\mathscr{P}, R)$ in the following way. The vertices of the graph are the elements of $\mathscr{P}$. We have an edge, which we call either $e(F, P)$ or $e(F', P')$, joining P and P' if and only if $R(F, P) = (F', P')$. So there is one edge for each face-pairing. Clearly, Connected$(\mathscr{P}, R)$ if and only if $\Gamma(\mathscr{P}, R)$ is connected.

Now let T be a maximal tree in $\Gamma(\mathscr{P}, R)$. Consider the equivalence relation $\sim$ on the disjoint union $\bigsqcup_{P \in \mathscr{P}} P$ of the elements of $\mathscr{P}$, generated by $x \sim A(F, P)(x)$ if $(F, P) \in \mathscr{F}(\mathscr{P})$, $e(F, P) \subset T$ and $x \in F$. We define the space $Y(\mathscr{P}, R, A, T)$ and the quotient map $\pi_Y : \bigsqcup_{P \in \mathscr{P}} P \to Y(\mathscr{P}, R, A, T)$ by identifying each equivalence class to a point. We have Connected$(\mathscr{P}, R)$ if and only if $Y(\mathscr{P}, R, A, T)$ is (arcwise) connected. Since in T no edge is a loop, all elements of $\mathscr{P}$ are naturally embedded in $Y(\mathscr{P}, R, A, T)$ — that is, the restriction to any component of the domain of the projection $\bigsqcup_{P \in \mathscr{P}} P \to Y(\mathscr{P}, R, A, T)$ is injective. It is straightforward to see that $Y(\mathscr{P}, R, A, T)$ is contractible if it is connected — a deformation retraction to a point can be constructed inductively, cell by cell, working along the edges of T.

For the rest of this section we will assume that Pairing$(\mathscr{P}, R, A)$, Connected$(\mathscr{P}, R)$ and Cyclic$(\mathscr{P}, R, A)$ are satisfied.

The following lemma is easy to prove.

LEMMA 4.1 (developing Y). *For any choice of* $P_0 \in \mathscr{P}$ *there exists a unique mapping* $D_Y : Y(\mathscr{P}, R, A, T) \to \mathbf{X}^n$, *which we call the developing map associated to* $(\mathscr{P}, R, A, T)$, *with the following properties:*

- $D_Y|_{P_0}$ *is the identity;*
- *for each* $P \in \mathscr{P}$, $D_Y|_P$ *is the restriction of an isometry of* $\mathbf{X}^n$ *(which we denote by* ψ_P*);*
- *if* $(F, P) \in \mathscr{F}(\mathscr{P})$ *and* $e(F, P) \subset T$ *joins* P *to* P', *then* $\psi_{P'} A(F, P) = \psi_P$.

A different choice of the initial convex cell P_0 *or a different choice of the way it is embedded in* $\mathbf{X}^n$ *leads to the mapping* $\psi \circ D_Y$ *for some* $\psi \in Isom(\mathbf{X}^n)$.

Changing the positions of the convex cells $P \in \mathscr{P}$ *(see Remark 3.1(a)), we may take each* ψ_P *to be the identity and then* $A(F, P)$ *is the identity for each edge* $e(F, P)$ *in* T.

From now on, we will assume that ψ_P is the identity for each $P \in \mathscr{P}$.

DEFINITION 4.2. We define an abstract group $G(\mathscr{P}, R, A, T)$ as the group generated by the set of symbols:

$$\{\alpha_{(F,P)} : (F, P) \in \mathscr{F}(\mathscr{P})\}$$

subject to the following relations:

- $\alpha_{(F,P)} = \mathrm{id}$ if $e(F, P) \subset T$.

- if $(F,P) \in \mathscr{F}(\mathscr{P})$, $e(F,P) \not\subset T$ and $R(F,P) = (F',P')$ then we have the relation $\alpha_{(F,P)}\alpha_{(F',P')} = \mathrm{id}$. In particular, if $R(F,P) = (F,P)$ then $\alpha_{(F,P)}$ has order two.
- for each $P_1 \in \mathscr{P}$ and for each codimension-two face C_1 of P_1, in the notation of Conditions 3.7 and 3.13, we have the relation

$$(\alpha_{(F_r',P_r)} \cdots \alpha_{(F_1',P_1)})^m = \mathrm{id}\ .$$

REMARK 4.3. According to Remark 3.14, given $P_1 \in \mathscr{P}$ and a codimension-two face C_1 of P_1, we obtain an equivalent relation starting from either of the codimension-one faces of P_1 containing C_1, or from any of the faces F_i or F_i'.

LEMMA 4.4 (holonomy). *We assume Pairing*$(\mathscr{P},R,A)$, *Connected*$(\mathscr{P},R)$ *and Cyclic*$(\mathscr{P},R,A)$. *For any choice of developing map* D_Y *associated to* $(\mathscr{P},R,A)$, *there exists a unique homomorphism* $h: G(\mathscr{P},R,A,T) \to \mathit{Isom}(\mathbf{X}^n)$ *with the following property: if* $(F,P) \in \mathscr{F}(\mathscr{P})$ *then* $h(\alpha_{(F,P)}) = A(F,P)$. *A different choice of* D_Y *leads to the homomorphism* $g \mapsto \psi h(g) \psi^{-1}$ *for some* $\psi \in \mathit{Isom}(\mathbf{X}^n)$.

Proof of 4.4. Given D_Y, the position in $\mathbf{X}^n$ of each $P \in \mathscr{P}$ is determined. For each $(F,P) \in \mathscr{F}(\mathscr{P})$, the face-pairing $A(F,P)$ is then also determined. We define $h(\alpha_{(F,P)}) = A(F,P)$. According to Pairing$(\mathscr{P},R,A)$ and to Lemma 4.1 the relations defining G starting from the generators $\alpha_{(F,P)}$ hold for the corresponding $A(F,P)$'s in Isom$(\mathbf{X}^n)$, and then h can be extended to a homomorphism of the whole of $G(\mathscr{P},R,A,T)$. Uniqueness is obvious. The last assertion is readily deduced from Lemma 4.1. □

DEFINITION 4.5. We abbreviate as follows: $Y = Y(\mathscr{P},R,A,T)$ and $G = G(\mathscr{P},R,A,T)$. We give G the discrete topology, and consider the space $G \times Y$ with the product topology. We consider on $G \times Y$ the equivalence relation $\sim$ generated by: $(g\alpha_{(F,P)},x) \sim (g, A(F,P)(x))$ whenever $g \in G, (F,P) \in \mathscr{F}(\mathscr{P})$ and $x \in F \subset P \hookrightarrow Y$. We will denote by $Z = Z(\mathscr{P},R,A,T)$ the quotient space of $G \times Y$ by this equivalence relation, and by $\pi_Z : G \times Y \to Z$ the quotient map.

REMARK 4.6 (Y not subset Z). It is false in general that the restriction to $\{\mathrm{id}\} \times Y$ of the projection $G \times Y \to Z$ is injective — see Example 4.12.

G acts on Z in an obvious way, and G acts on $\mathbf{X}^n$ via the homomorphism h.

LEMMA 4.7 (developing Z). *For any choice of* P_0 *in* $\mathscr{P}$, *there exists a unique G-equivariant mapping* $D_Z : Z \to \mathbf{X}^n$ *such that the element of* Z *represented by* $(g, y) \in G \times Y$ *is mapped to* $h(g)(D_Y(y))$, *where* D_Y *and* h *are given respectively by Lemma 4.1 and Lemma 4.4. A different choice of the initial convex cell and its position in* $\mathbf{X}^n$ *leads to the mapping* $\psi \circ D_Z$ *for some* $\psi \in Isom(\mathbf{X}^n)$.

Proof of 4.7. We only have to check that if $(g, y) \sim (g', y')$ in $G \times Y$ then $h(g)(D_Y(y)) = h(g')(D_Y(y'))$, and this is readily deduced from the definition of $\sim$ and from the definitions of D_Y and h. $\square$

COROLLARY 4.8 (P embeds in Z). *For each* $P \in \mathscr{P}$ *and* $g \in G$ *the mapping*

$$P \ni x \mapsto \pi_Z(g, x) \in Z$$

is injective.

REMARK 4.9 (Z independent of T). The definition of Z given above depends on T. However, this dependence is not real. To see this we define $\mathscr{G}$ to be a groupoid (a small category in which every morphism has a two-sided inverse). We take $\mathscr{P}$ to be the set of objects of $\mathscr{G}$. We take $\alpha_{(F,P)}$ to be a morphism from P to P', where $R(F, P) = (F', P')$. In general, the morphisms are formed from compositions of these, subject to the same relations as those used in the definition of G above, except that we now take $T = \varnothing$. We give the set M of morphisms of $\mathscr{G}$ the discrete topology, and we take the obvious topology on $\bigsqcup_{P \in \mathscr{P}} P$. To define Z, we fix $P_0 \in \mathscr{P}$, and let $M(P_0)$ be the set of morphisms with range P_0. We then take all pairs $(g, x) \in M(P_0) \times \bigsqcup_{P \in \mathscr{P}} P$, where $x \in P$, $P \in \mathscr{P}$ and $g : P \to P_0$. We identify $(g\alpha_{(F,P)}, x)$ with $(g, A(F, P)x)$, provided $x \in F \subset P$ and $g : P' \to P_0$, where $R(F, P) = (F', P')$. We define Z to be the identification space just defined. If we change P_0, the resulting Z is unchanged. An isomorphism between the two versions of Z is given by choosing a word in the $\alpha_{(F,P)}$'s relating the choices. The isomorphism is therefore determined up to the action of an element of M.

The only reason for using the definition given previously, in terms of a group, rather than that given now, is that the concept of a group is more familiar than the concept of a groupoid. The construction of the group G from the groupoid $\mathscr{G}$ is the standard construction of a group from a connected groupoid. We are therefore justified in writing $Z(\mathscr{P}, R, A)$ instead of $Z(\mathscr{P}, R, A, T)$, if the occasion demands.

REMARK 4.10 (cell structure of Z). For each $g \in M(P_0)$ and each face E of P, we obtain the subset $g(E) = \pi_Z(\{g\} \times E)$ of Z. To see that $g(E)$ is an isomorphic copy of E, we apply the developing map D_Z. So $g(E)$ is a convex cell of the same dimension as E. Since the identifications respect the face structure (see Condition 3.2(c)), we see that Z is the disjoint union of the relative interiors of these convex cells of various dimensions. Of course, g and E are not determined by the cell; $g(E)$ is just one representation. The left action of G preserves the cell structure of Z. If x and y are interior points of the same top-dimensional cell of Z and if $gx = y$ for some $g \in G$, then $x = y$ and g is the identity element.

It is easy to see that Connected($\mathscr{P}$, R) is equivalent to Z being (arcwise) connected.

DEFINITION 4.11 (boundary and interior of Y). We write $Y = Y(\mathscr{P}, R, A, T)$. The *boundary of* Y, denoted ∂Y, is the union of the faces F such that $(F, P) \in \mathscr{F}(\mathscr{P})$ and $e(F, P) \not\subset T$. The *interior of* Y is the complement of the boundary.

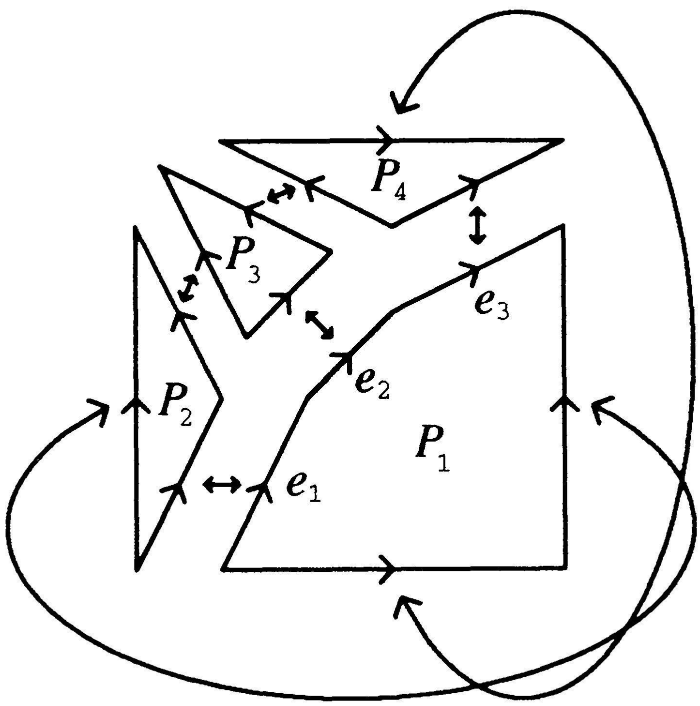

FIGURE 10.
Face pairings.
A set of polyhedra in Euclidean two-space, and a description of their face-pairings.

EXAMPLE 4.12 (fundamental domain not embedded). Let $\mathscr{P}$ be the set of polyhedra in $\mathbf{E}^2$ shown in Figure 10, and let the face-pairing (R, A) be defined by the arrows in the picture, in such a way that the orientation of the edges is preserved. All the conditions described in Section 3 hold for $(\mathscr{P}, R, A)$. It is evident from Figures 10 and 11 the developing map $D_Y: Y \to \mathbf{E}^2$ is not injective.

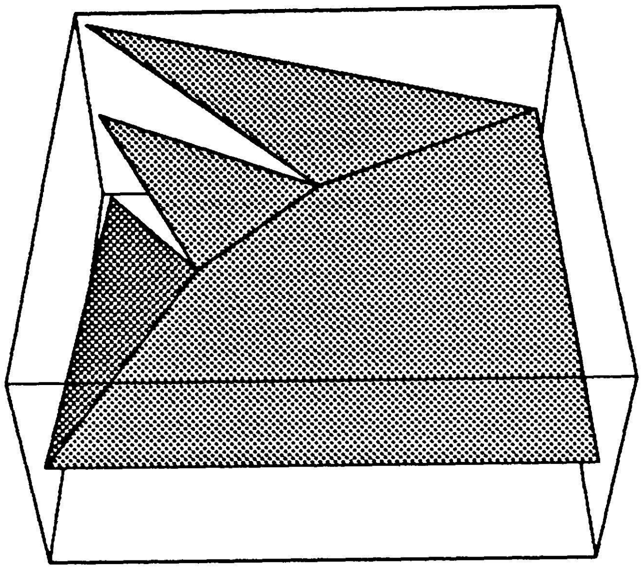

FIGURE 11.
The space Y.
We illustrate the space Y arising from Figure 10.

THEOREM 4.13 (modelled on $\mathbf{X}^n$). *Let $n \geqslant 2$, let $\mathscr{P}$ be a set of thick convex cells in $\mathbf{X}^n$ and let (R, A) be a face-pairing such that:*

(a) *Pairing$(\mathscr{P}, R, A)$;*

(b) *Connected$(\mathscr{P}, R)$;*

(c) *Cyclic$(\mathscr{P}, R, A)$;*

(d) *LocallyFinite$(\mathscr{P}, R, A)$ (recall from Condition 3.29 that this condition is automatically true if $n = 2$).*

Let T be a maximal tree in $\Gamma(\mathscr{P}, R)$, set $Y = Y(\mathscr{P}, R, A, T)$, $G = G(\mathscr{P}, R, A, T)$ and $Z = Z(\mathscr{P}, R, A, T)$, and let $D_Y: Y \to \mathbf{X}^n$, $h: G \to Isom(\mathbf{X}^n)$, $D_Z: Z \to \mathbf{X}^n$ be the developing maps as in Lemma 4.1, Lemma 4.4 and Lemma 4.7. Then Z is endowed with an $\mathbf{X}^n$-structure with respect to which $D_Z: Z \to \mathbf{X}^n$ is a local isometry. Also the convex cell structure of Z (see Remark 4.10) is locally finite. Furthermore the action of the group G on Z is proper discontinuous. Let p be a point in the interior of a top-dimensional cell P of Z. Then the stabilizer of p is trivial, and the orbit of p contains no other point of P.

This result will be proved by induction on n, assuming the following result in dimensions less than n. In Section 5, we will complete the induction by showing how Theorem 4.13 in dimension n implies Theorem 4.14 in dimension n.

THEOREM 4.14 (Poincaré's Theorem Version 1). *Let* $n \geqslant 2$, *let* $\mathscr{P}$ *be a set of thick convex cells in* $\mathbf{X}^n$ *and let* (R, A) *be a face-pairing such that:*

(a) *Pairing*$(\mathscr{P}, R, A)$;

(b) *Connected*$(\mathscr{P}, R)$;

(c) *Cyclic*$(\mathscr{P}, R, A)$;

(d) *Metric*$(\mathscr{P})$.

Let T *be a maximal tree in* $\Gamma(\mathscr{P}, R)$, *set*

$$Y = Y(\mathscr{P}, R, A, T), \quad G = G(\mathscr{P}, R, A, T), \quad Z = Z(\mathscr{P}, R, A, T)$$

and let $D_Y : Y \to \mathbf{X}^n$, $h : G \to \mathit{Isom}(\mathbf{X}^n)$, $D_Z : Z \to \mathbf{X}^n$ *be mappings as in 4.1, 4.4 and 4.7. Then the following conclusions hold:*

(e) *LocallyFinite*$(\mathscr{P}, R, A)$ *is true in its strong form (see Remark 3.31);*

(f) Z *is endowed with an* $\mathbf{X}^n$*-structure with respect to which* $D_Z : Z \to \mathbf{X}^n$ *is a (bijective) isometry;*

(g) $h : G \to \mathit{Isom}(\mathbf{X}^n)$ *is injective and its image is a discrete subgroup of* $\mathit{Isom}(\mathbf{X}^n)$;

(h) $D_Y : Y \to \mathbf{X}^n$ *is injective on the interior of* Y *(see Definition 4.11), so that* $D_Y(Y)$ *or its interior can be considered as a fundamental domain for the action of* $h(G)$ *on* $\mathbf{X}^n$, *depending on the precise definition of that concept;*

(j) *the convex cell structure of* Z *(see Remark 4.10) is locally finite.*

The hypotheses (and hence the conclusions) hold in particular if we add to conditions *Pairing*$(\mathscr{P}, R, A)$, *Connected*$(\mathscr{P}, R)$ *and* *Cyclic*$(\mathscr{P}, R, A)$ *either of the following additional conditions:*

(k) $\mathbf{X}^n = \mathbf{E}^n$ *or* $\mathbf{S}^n$ *and* *Finite*$(\mathscr{P})$;

(l) $\mathbf{X}^n = \mathbf{H}^n$, *Finite*$(\mathscr{P})$ *and* *FirstMetric*$(\mathscr{P})$;

Proof of 4.13. We will assume that Theorem 4.14 has been proved in dimensions less than n.

For $n = 2$, Theorem 4.13 is a consequence of Condition 3.19. To see this, note that each point in $\pi_Z^{-1}(z)$ lies in some C_j $(1 \leqslant j \leqslant r)$ of one particular cycle, where the notation comes from Condition 3.10. Let $m > 0$ be as in Condition 3.13. *A priori,* we do not know that there are m distinct copies in Z of each of the r dihedral regions at the various $C_j \subset P_j$, though we do know that there are no more than these mr regions around $z \in Z$, because of the way Z is constructed. The existence of D_Z shows that there are also no fewer than mr regions. This completes the proof for $n = 2$.

We now prove Theorem 4.13 for $n > 2$, assuming Theorem 4.14 in dimensions less than n. Let $\pi_Z(g, x) = z$, and let $x \in \mathrm{RelInt}(E)$, where E is a face of $P \in \mathscr{P}$ of codimension i. Since we are assuming LocallyFinite$(\mathscr{P}, R, A)$, we have a finite graph Γ_E which is the component of $\Gamma^i(\mathscr{P}, R, A)$ containing (E, P) as a vertex (see Definition 3.28). Each vertex of Γ_E is a pair of the form (E', P'), where E' is a codimension-i face of $P' \in \mathscr{P}$. The link (see Definition 2.14) of E' in P' is a convex cell in $\mathbf{S}^{n-1}$, which is well-defined up to isometry.

Let $\mathscr{P}_E$ be the finite collection of links arising from the finite set of vertices of Γ_E. These are convex polyhedra in $\mathbf{S}^{n-1}$, defined up to isometry. For each vertex (E', P') of Γ_E, we choose a point $u' \in \mathrm{RelInt}(E')$. We make no attempt to choose these points consistently — indeed, in general consistency of choice is not possible. The position of a link in $\mathbf{S}^{n-1}$ is determined by fixing an isomorphism between $\mathbf{R}^n$ and the tangent space at u'. The given face-pairing (R, A) induces a face-pairing (R_E, A_E) on $\mathscr{P}_E$ as follows. Suppose E' is a face of F', $R(F', P') = (F'', P'')$, and $A(F', P')$ is the corresponding face-pairing. Let $E'' = A(F', P')(E')$. Let $u' \in \mathrm{RelInt}(E')$ and $u'' \in \mathrm{RelInt}(E'')$ be the points we have chosen. We define the face-pairing $A_E(E', F', P')$ by applying $A(F', P')$ to the tangent space at u', and then parallel translating from $A(F', P')(u')$ to u''. This definition of the face-pairing is clearly independent (in the appropriate sense) of the choice of the points u' and u''. It is easy to check the truth of Pairing$(\mathscr{P}_E, R_E, A_E)$.

Connected$(\mathscr{P}_E, R_E)$ follows from the connectedness of Γ_E. Cyclic$(\mathscr{P}_E, R_E, A_E)$ follows immediately from Cyclic$(\mathscr{P}, R, A)$. Metric$(\mathscr{P}_E)$ follows from Remark 3.24, applied to $\mathscr{P}_E$. We apply Theorem 4.14 in dimension $n - 1$ to deduce that the developing map $Z(\mathscr{P}_E, R_E, A_E) \to \mathbf{S}^{n-1}$ is an isometry. The induction also tells us that the cell structure of $Z(\mathscr{P}_E, R_E, A_E)$ is finite.

We choose $z \in \mathrm{RelInt}(E)$, and identify E with $\mathrm{id}(E)$, in the notation of Remark 4.10. Each cell of $Z(\mathscr{P}_E, R_E, A_E)$ corresponds to a triple of the form (h, E', P') where h is a member of the finite groupoid $\mathscr{G}(\mathscr{P}_E, R_E, A_E)$. The face-pairings identify z with the point $\pi_Z(z')$, where $z' \in \mathrm{RelInt}(E')$ depends on (h, E', P'). Since the setup is finite, we can choose $\delta > 0$ simultaneously for all (h, E', P') so that the only faces of P' met by the δ-neighbourhood centred at z' are those that contain E'.

There is a map of $Z(\mathscr{P}_E, R_E, A_E)$ into the δ-neighbourhood of z in $Z(\mathscr{P}, R, A)$, since each of the groupoid relations relevant in the definition of the first space will also apply to the second. Any identification of a point of $\mathrm{RelInt}(E)$ with another point, when $Z(\mathscr{P}, R, A)$ is formed, can only be formed as a result of face-pairings which are also in (R_E, A_E). Thus the strong form of local finiteness (see 3.31) is satisfied by $(\mathscr{P}, R, A)$.

The composition of $D_Z : Z(\mathscr{P}, R, A) \to \mathbf{X}^n$ with the obvious map from $Z(\mathscr{P}_E, R_E, A_E)$ to $Z(\mathscr{P}, R, A)$ can be identified with the developing map $Z(\mathscr{P}_E, R_E, A_E) \to \mathbf{S}^{n-1}$ by a change of scale in the range. By induction, this developing map is an isometry. Therefore the obvious map of $Z(\mathscr{P}_E, R_E, A_E)$ to $Z(\mathscr{P}, R, A)$ is injective and the image of $Z(\mathscr{P}_E, R_E, A_E)$ is mapped injectively by D_Z. It follows easily that a neighbourhood of z in $Z(\mathscr{P}, R, A)$ is the cone on $\mathbf{S}^{n-1}$, which is mapped isometrically to $\mathbf{X}^n$ by D_Z. □

The main part of the induction step for Theorem 4.14 will be proved in Section 5. At this point, we prove only a small part of this result.

LEMMA 4.15 (locally finite). *LocallyFinite$(\mathscr{P}, R, A)$ follows from the hypotheses of Theorem 4.14 and the inductive hypothesis that Theorem 4.14 is true in dimensions less than n.*

Proof of 4.14. In the proof of Theorem 4.13 we used Locally-Finite$(\mathscr{P}, R, A)$ in order to show that the link of z is embedded in $Z(\mathscr{P}, R, A)$ and that the local picture is as we expect. Here we are trying to prove LocallyFinite$(\mathscr{P}, R, A)$, so the argument needs to be modified. Note that Metric$(\mathscr{P})$, which we are now assuming, implies SecondMetric$(\mathscr{P})$, which in turn implies Metric$(\mathscr{P}_E)$.

The version of Theorem 4.14 for $\mathbf{S}^{n-1}$ is already known inductively, and so we know that $Z(\mathscr{P}_E, R_E, A_E) = \mathbf{S}^{n-1}$. We deduce that the tessellation of $Z(\mathscr{P}_E, R_E, A_E)$ is finite. This means that we have proved the strong form of LocallyFinite$(\mathscr{P}, R, A)$ (see 3.31). □

5. DEFINING A METRIC

If Pairing$(\mathscr{P}, R, A)$ and Connected$(\mathscr{P}, R)$, we obtain the connected quotient space $Q = Q(\mathscr{P}, R, A)$ defined in Remark 3.6. We can define a "metric" on Q in the obvious way: Given two points z_1 and z_2 in Q, we join them with a special kind of path in Q. The path is divided into a finite number of subpaths, and each subpath is the image of a rectifiable path in some $P \in \mathscr{P}$. The distance between z_1 and z_2 is defined as the infimum over all such paths of the sum of the lengths of the subpaths. We get the same infimum if we restrict to subpaths starting and ending in the interior of a codimension-one