8. SPECIAL CASES

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matrices each of which is an isometry of $\mathbf{X}^n$, and such that each entry is an algebraic number. We can hold the algebraic numbers in the computer by holding the coefficients of its irreducible polynomial, together with a floating point approximation to the number. Suppose we are also given a finite set of finite-sided polyhedra, given approximately using floating point numbers, together with face-pairings each of which is equal to one of our given matrices. We can then check the condition Cyclic$(\mathscr{P}, R, A)$ precisely, using integer arithmetic, by checking on a certain product of face-pairings. (We can use floating point arithmetic to see which words in the face-pairings need to have checks performed.)

8. Special cases

One case of Poincaré's Theorem which is often used is the case where there is a single element of $\mathscr{P}$ and all face-pairings are reflections. In that case completeness is a consequence of Lemma 5.4, provided the other axioms are satisfied. This enables a number of important examples to be constructed.

As a minor point, we note that it enables us to construct infinitely generated fuchsian groups with an arbitrary subset of the positive integers being the set of exponents of maximal cyclic subgroups. These and other applications of Poincaré's Theorem are well-known.

Poincaré's Theorem works in an especially simple way in dimension two. In this dimension, a face-pairing is called an *edge-pairing*. The following result is essentially due to de Rham [dR71].

THEOREM 8.1 (dimension two). *Suppose we have a finite set $\mathscr{P}$ of finite-sided polygons in $\mathbf{H}^2$ and an edge-pairing (R, A) of the boundary edges satisfying Pairing$(\mathscr{P}, R, A)$, Connected$(\mathscr{P}, R)$ and Cyclic$(\mathscr{P}, R, A)$. Then the quotient Q of $\bigsqcup_{P \in \mathscr{P}} P$ by the edge-pairing is a two-dimensional hyperbolic orbifold which is obtained from a complete orbifold with geodesic boundary by removing the compact boundary components. The hyperbolic structure on Q is induced in an obvious way from the hyperbolic structure on the hyperbolic polygons used to define it. The group G generated by the edge-pairings in the manner described in Definition 4.2 is discrete. If all the polygons are compact, then Q is a compact orbifold without boundary. (But it may have mirrors.)*

REMARK 8.2. The main feature of this result is that for the two-dimensional case it describes the quotient Q even when this is not complete. (For the conditions under which Q is complete the reader is referred to Lemma 5.4 and Theorem 6.3.)

REMARK 8.3. As an orbifold, the boundary of Q is empty, since each edge is paired to some other. If an edge is paired to itself by a reflection, Q has a corresponding mirror. This is a (possibly non-compact) boundary component of the underlying manifold, but not an orbifold boundary component. The completion of Q may well have orbifold boundary components which are not in Q itself: each of these is a circle.

Proof of 8.1. We first look at each ideal point p which is the end of two distinct boundary components of some $P \in \mathscr{P}$ (in this situation we will say p is a peak of P). We remove from P a small horodisk neighbourhood centred at p. If we glue the remaining pieces together using the edge-pairings, then the boundary horocycles do not necessarily match up and we obtain a hyperbolic orbifold T whose boundary is a union of topological circles and arcs, each of which is a finite union of horocyclic and geodesic arcs.

We now need to glue back the pieces we have cut out. We first glue together the horodisk pieces corresponding to a single boundary component of T, obtaining an orbifold B_i, and understand what B_i looks like. This can be done by using the horocyclic foliation of B_i.

Each piece constituting B_i is triangular, where two of the sides are geodesic rays which are asymptotic and one side is a horocyclic arc. When we glue these pieces together, several things can happen.

(a) The pieces glue in a cyclic fashion and the associated holonomy is parabolic. Then B_i is complete. Gluing B_i into place gives rise to a cusp in Q.

(b) The pieces glue in a cyclic fashion and the associated holonomy is hyperbolic. Then the developing image into the upper half-plane is as shown in Figure 13. In this case B_i is a cylinder, which at one end is incomplete, with the completion adding a compact geodesic, and at the other end is bounded by alternate horocyclic and geodesic arcs. (Actually, a suitable choice of the pieces at the beginning allows to reduce to the case of one geodesic arc and one horocyclic arc.) Gluing B_i into place gives an incomplete end for Q. The completion adds to Q a compact geodesic boundary component.

(c) The pieces do not glue together in a cyclic fashion. This means that in both directions one eventually reaches a geodesic ray which is glued either to itself (by a reflection) or to a geodesic ray whose point at infinity is not a peak.

As a set, B_i is identified to a triangle in $\mathbf{H}^2$ with one vertex at infinity, two sides which are asymptotic geodesic half-lines and the third

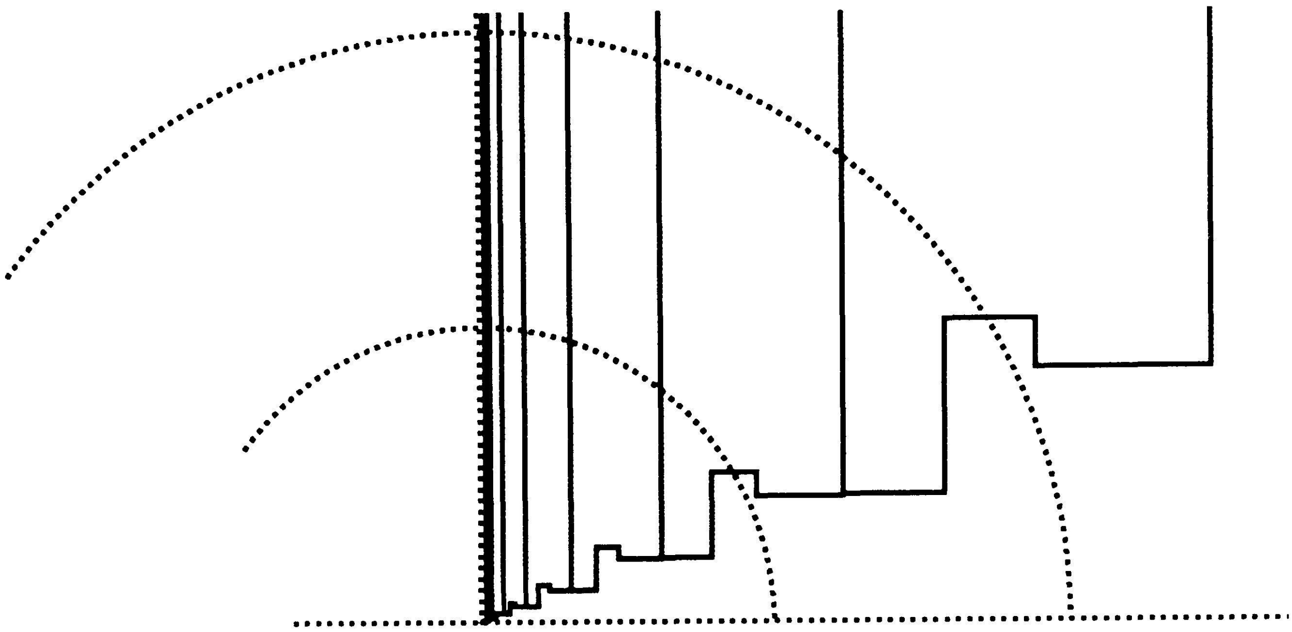

FIGURE 13.

Hyperbolic holonomy.

This picture shows the developing image associated with B_i in the situation described in (b) in the proof of 8.1. The dotted lines show an alternative fundamental domain which shows the structure of B_i as a cylinder more clearly.

side which consists of alternate horocyclic and geodesic arcs (a suitable choice of the pieces at the beginning actually allows to obtain only one horocyclic arc and no geodesic arcs). As a hyperbolic orbifold, B_i is either isomorphic to the triangle or obtained from the triangle by assuming that one or both the geodesic half-lines are mirrors. In particular, B_i is complete. The situation is shown in Figure 14.

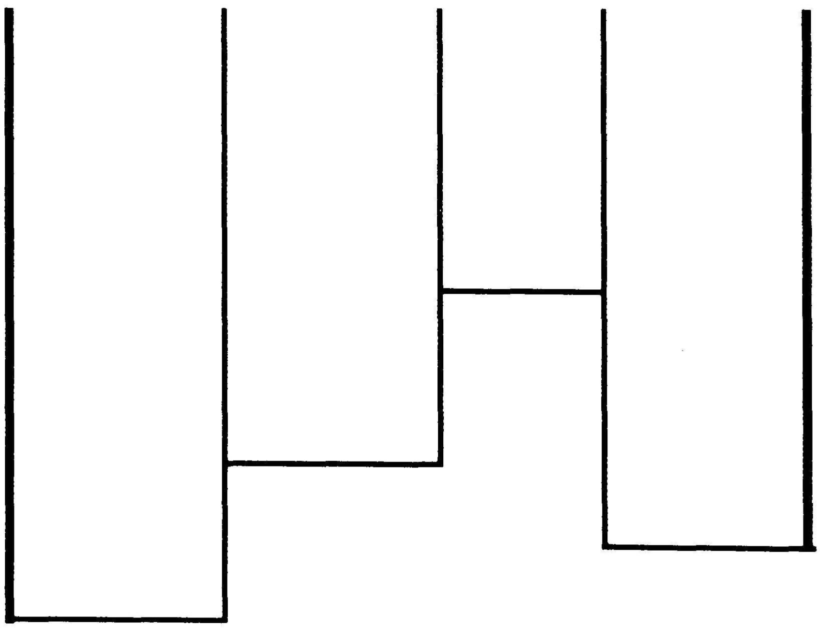

FIGURE 14.

Mirrors.

One or both the vertical thick edges may represent mirrors, as described in (c) in the proof of 8.1.

A consistent horocycle always exists in this case, so gluing B_i into place gives us a metric in Q which is complete near B_i. We can also describe what Q looks like near B_i. If we call wedge a region in $\mathbf{H}^2$ bounded by two geodesic half-lines with common origin, then a subset of Q which contains B_i is obtained as follows: for every half-infinite geodesic side s of B_i which is not a mirror, glue a suitably thin wedge to B_i by identifying s with one of the sides of the wedge.

This description of the possible situations implies the conclusion of the proof. □

9. Literature review

It seems to the authors that a minimal requirement for a satisfactory treatment of Poincaré's Theorem is that it should apply directly to the case of a finite-sided Dirichlet domain resulting from the action of a discontinuous group of isometries on one of the three constant curvature geometries $\mathbf{S}^n$, $\mathbf{E}^n$ and $\mathbf{H}^n$. Furthermore the hypotheses should be easy to verify, and extraneous hypotheses should not be included. We review the literature with these criteria in mind.

The first versions of Poincaré's Theorem were published in [Poi82], covering the two-dimensional version, and [Poi83], covering the three-dimensional version. These are reprinted in Volume Two of [Poi52]. It is clear that Poincaré understood very well what was going on. However, the papers are not easy to read. In particular, the reader of the three-dimensional case is referred to the treatment of the two-dimensional case for proofs; this is fully acceptable for a trail-blazing paper, but not satisfactory in the long term.

There are a number of reasonable published versions of Poincaré's Theorem in dimension two. Of these, we would single out the version by de Rham [dR71] as being particularly careful and easy to read. Most published versions of Poincaré's Theorem applying to all dimensions are unsatisfactory for one reason or another. The most satisfactory version is [Sei75], due to Seifert. The proofs are careful and rigorous, but rather long. Poincaré's Theorem is proved in all dimensions and for all three constant curvature geometries. The treatment is not constructive in several aspects, specially when it comes to completeness. There is some discussion of conditions which are equivalent to completeness in the hyperbolic case, which are closer to being constructive. However this discussion is limited to the finite volume case. Seifert's treatment also contains unnecessary restrictions, which, for example,