9. Literature review

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A consistent horocycle always exists in this case, so gluing B_i into place gives us a metric in Q which is complete near B_i. We can also describe what Q looks like near B_i. If we call wedge a region in $\mathbf{H}^2$ bounded by two geodesic half-lines with common origin, then a subset of Q which contains B_i is obtained as follows: for every half-infinite geodesic side s of B_i which is not a mirror, glue a suitably thin wedge to B_i by identifying s with one of the sides of the wedge.

This description of the possible situations implies the conclusion of the proof. □

9. Literature review

It seems to the authors that a minimal requirement for a satisfactory treatment of Poincaré's Theorem is that it should apply directly to the case of a finite-sided Dirichlet domain resulting from the action of a discontinuous group of isometries on one of the three constant curvature geometries $\mathbf{S}^n$, $\mathbf{E}^n$ and $\mathbf{H}^n$. Furthermore the hypotheses should be easy to verify, and extraneous hypotheses should not be included. We review the literature with these criteria in mind.

The first versions of Poincaré's Theorem were published in [Poi82], covering the two-dimensional version, and [Poi83], covering the three-dimensional version. These are reprinted in Volume Two of [Poi52]. It is clear that Poincaré understood very well what was going on. However, the papers are not easy to read. In particular, the reader of the three-dimensional case is referred to the treatment of the two-dimensional case for proofs; this is fully acceptable for a trail-blazing paper, but not satisfactory in the long term.

There are a number of reasonable published versions of Poincaré's Theorem in dimension two. Of these, we would single out the version by de Rham [dR71] as being particularly careful and easy to read. Most published versions of Poincaré's Theorem applying to all dimensions are unsatisfactory for one reason or another. The most satisfactory version is [Sei75], due to Seifert. The proofs are careful and rigorous, but rather long. Poincaré's Theorem is proved in all dimensions and for all three constant curvature geometries. The treatment is not constructive in several aspects, specially when it comes to completeness. There is some discussion of conditions which are equivalent to completeness in the hyperbolic case, which are closer to being constructive. However this discussion is limited to the finite volume case. Seifert's treatment also contains unnecessary restrictions, which, for example, would prevent his version being directly applicable to the Dirichlet domain applied to a rotation through π about a fixed point in dimension two. (One would first have to subdivide the boundary of the Dirichlet domain, since Seifert assumes that the map to the quotient space is injective on each face of the given polyhedron.)

The treatment in [Mas88] is difficult to understand. For example in H.9 on page 75, it is claimed that a metric is defined in a certain way, and this fact is said to be "easy to see", but it seems to us an essential and non-trivial point, which is not so easy to see, particularly when the group generated by the face-pairings is not discrete. Maskit's proof does not use induction on dimension, which seems to us essential for a simple and clear treatment. We refer in particular to the assertions that certain maps are homeomorphisms on page 77. The Proposition in IV.1.6 on page 79 of this book is incorrect — a counter-example is given in Example 9.1 — because there are no infinite cycles or infinite edges according to the definitions in the book. As in the case of Seifert's paper, the constructive aspect is ignored, and the question of completeness is handled in an entirely non-constructive way. Maskit's local finiteness condition is more demanding than ours, and Seifert's is more demanding than Maskit's.

In [Ril83], there is a statement of Poincaré's Theorem with no proof, and [Sei75] is cited. Unfortunately, Riley fails to take into account Seifert's restriction to the finite volume case. This leads him to a statement of Poincaré's Theorem, which implies that if two parallel vertical planes in upper half-space are matched by a hyperbolic isometry, then the infinite cyclic group thus generated is discrete.

Maskit's paper [Mas71] contains a nice discussion of completeness, though again it is not a constructive approach. He limits his discussion to hyperbolic space in dimensions two and three. We are not confident that the arguments in the paper are complete. For example, there seems to be an assumption that the quotient of a metric space, such that the inverse image of any point is finite, is again metric. This is false, as is shown by identifying x with $-x$ in $[-1, 1]$, provided $0 \leqslant x < 1$. A slight variation of this gives a counter-example in which the inverse image of a point is always equal to two points.

EXAMPLE 9.1 (incomplete example). Take a quadrilateral in the euclidean plane with no two sides parallel, and multiply with $(0, \infty)$. Embed this in the upper half-space model of $\mathbf{H}^3$, with the quadrilateral embedded in a horizontal horosphere, and the factor $(0, \infty)$ corresponding to vertical straight lines. This gives us a convex hyperbolic polyhedron P with four faces.

The quadrilateral gives rise to two commuting orientation-preserving euclidean similarities which identify opposite sides. These similarities can be regarded as hyperbolic isometries which are face-pairings for P. They do not generate a discrete group of isometries of $\mathbf{H}^3$. Maskit's paper [Mas71] and his book [Mas88] both contain statements implying that this group of isometries is discrete.

There is a discussion of Poincaré's Theorem in Beardon's paper [Bea83]. Beardon concentrates on $\mathbf{H}^2$, with a single compact convex polygon. Questions of completeness are not treated.

Morokuma's paper [Mor78] is another paper which is difficult to read. If the definitions in this paper are taken literally, then the statement of the main theorem implies that a closed ball of finite radius is equal to the whole of hyperbolic space. This is because a closed ball is the intersection of a collection of half-spaces, each containing the ball in its interior, and as a consequence a closed ball is a polyhedron with no faces. The paper contains a great deal of notation, which, to our way of thinking, obscures the ideas. On occasion the author seems to assume the main point of what needs to be proved. For example, on page 163 of his article, the statement "$\tau^{-1}p' \in F'_{k+1}$ namely $F'_{k+1} = F'$" would not be true if Morokuma's group Γ were not discrete. But at this point he is trying to prove discreteness.

Apanasov's paper [Apa86] is yet another paper which is difficult to read. Apanasov allows non-convex polyhedra. To see the consequences of Apanasov's definitions, consider the Poincaré disk model for $\mathbf{H}^2$. According to his definitions, the union of the closed first and third quadrants is a polyhedron with two one-dimensional faces, namely the x- and y-axes. There are no codimension-two faces. The intersection of two faces of a polyhedron does not need to be a face. It is not clear to us what is meant by Condition IV on page 474 of the English translation of Apanasov's paper. As a general comment on this paper, it seems as though much of what one should prove in Poincaré's Theorem are presented as hypotheses, rather than as conclusions.

An earlier paper by Aleksandrov, [Ale54], also makes many parts of Poincaré's Theorem into hypotheses rather than conclusions.

A proof of Poincaré's theorem in the special case of a single polyhedron with each face-pairing equal to the reflection in that face is given in [dlH91]; this proof has the same inductive structure as the proof given in our paper. The only condition to check is that the angles at codimension 2 faces have the form π/m for some integer m. This version of Poincaré's theorem is readily deduced from Theorem 5.5 using 5.4; in fact LocallyFinite is obvious in this case and the quotient space is complete as it is identified with the polyhedron itself.