10. Appendix

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This appendix contains results due to Brian Bowditch, published here with his permission.

We recall that a finite-sided closed convex cell of $\mathbf{H}^{n+1}$ is said to be *pyramidal* at an ideal point p if any two faces whose closures contain p meet in $\mathbf{H}^{n+1}$. The intersection of such a convex cell with a horosphere centred at p is a euclidean finite-sided closed convex cell of dimension n (provided the horosphere only meets faces which have p as an ideal point). One way to see this is to use the upper half-space model with p equal to the point at infinity. Conversely, given a convex finite-sided n-dimensional euclidean cell, we can think of this cell as lying in a horosphere which is a horizontal subspace in the upper half-space model. This gives rise to an $(n+1)$-dimensional hyperbolic convex cell, by taking the intersection of vertical half-spaces determined by the half-spaces defining the euclidean convex cell. We use the names "pyramidal" and "non-pyramidal" for convex euclidean cells if the corresponding hyperbolic cells are pyramidal or non-pyramidal respectively. A euclidean convex cell is non-pyramidal if and only if it has disjoint faces. If a euclidean cell is pyramidal, then there is a face which is the intersection of all other faces, that is there is a unique minimal face. A pyramidal euclidean n-cell is the product of an i-dimensional cell with the cone on a spherical $(n-i-1)$-dimensional cell. (The cone point is placed at the centre of the $(n-i-1)$-dimensional sphere.)

Let M be a connected euclidean similarity n-dimensional manifold which is the union of a locally finite set of closed subsets $\{X_i\}$. Each X_i has an induced similarity structure which is isomorphic to that of a closed finite-sided euclidean convex polyhedron. There are only a finite number of distinct similarity classes of X_i. The intersection of any face of any X_i with any face of any X_j is a common face of each. This implies that M has the structure of a locally finite polyhedral cell complex. Let G be a group of similarities of M which preserve the cell structure. Suppose that the number of orbits of non-pyramidal polyhedra is finite.

Theorem 10.1 (Bowditch). *Under the above assumptions, the number of orbits of cells is finite. Moreover, the number of orbits is bounded in terms of the number of orbits of non-pyramidal cells and the geometry (up to similarity) of the* X_i.

Bowditch has suggested that if there is one or more pyramidal polyhedral cell, then one should be able to prove that G is a finite group. It would follow that G consists of euclidean isometries and that M contains only a finite number of cells. This conjecture remains open.

Proof of 10.1. Let X be the union of the non-pyramidal cells in M, and let Y be the union of cells which meet X. Note that $X \subset Y$.

Now suppose there is a top-dimensional cell which is not in Y and let σ be its unique minimal face. Then σ is similar to $\mathbf{R}^i$ for some i. If α is any cell meeting σ, then $\sigma \subset \alpha$ since σ is minimal. Clearly α is not in X. Therefore σ is the unique minimal face of α. We have seen above that α is the product of σ and the cone on a convex subset $\mathbf{S}^{n-i-1}$. It follows that the union of the cells meeting σ is the product of σ with the cone on $\mathbf{S}^{n-i-1}$. It follows that the cell structure of M is finite, G is a finite group and $X = \varnothing$. The other possibility is that $Y = M$.

Let $K \subset X$ be a finite union of cells such that $GK = X$. The cell structure of M is locally finite, with a bound for the number of cells in any small neighbourhood being given by the geometry of the X_i. The number of cells of M which meet K is bounded by the number of cells of K and the maximum possible number of cells meeting a fixed small neighbourhood of any fixed point of K. This gives an upper bound for the number of orbits of cells of M under the action of G in case $Y = M$. If $X = Y = \varnothing$, then the number of cells of M is bounded by the geometry of the X_i. □

We apply Theorem 10.1 to find out a litte more about the spaces that arise in Poincaré's Theorem. Suppose the hypotheses Pairing($\mathscr{P}, R, A$), Connected($\mathscr{P}, R$), Finite($\mathscr{P}$) and Cyclic($\mathscr{P}, R, A$) are satisfied for a set of convex cells (see Definition 2.8) in $\mathbf{H}^n$. To each convex cell we adjoin the ideal points, so as to obtain a compact space. The face-pairings are defined on the closures of the faces. Let $\bar{Q}$ be the quotient of the disjoint union of the extended cells by the face-pairings, endowed with the quotient topology.

THEOREM 10.2. *$\bar{Q}$ is a compact hausdorff space.*

Proof of 10.2. Let X be the disjoint union of the closures of the convex cells. So X is compact and hausdorff. We first show that the inverse image of a point under the quotient map $X \to \bar{Q}$ is a finite set. This is clear from Theorem 4.13 for any point which is not an ideal point. For an ideal point p, we can construct a similarity manifold to which Theorem 10.1 applies, by developing a horosphere centred at p into $\mathbf{R}^{n-1}$. More details, which will help the interested reader with the construction of the similarity manifold, are given in the discussion of Definition 6.2.

A pyramidal cell in $\mathbf{R}^{n-1}$ corresponds to a convex cell in $\mathbf{H}^n$ together with an ideal point p in its boundary, such that any two faces with closures containing p meet inside $\mathbf{H}^n$. A non-pyramidal cell corresponds to a convex cell in $\mathbf{H}^n$ and an ideal point p contained in the closures of two non-intersecting faces of the convex cell. The hypothesis needed in order to apply Theorem 10.1, that there are only a finite number of orbits of non-pyramidal cells, comes from the fact that there are only a finite number of pairs of faces and therefore only a finite number of pairs of non-intersecting faces which meet at infinity.

It follows that the inverse image in X of any point of $\bar{Q}$ is finite. Moreover the number of points in the inverse image is bounded by a fixed integer N. Two points $x, y \in X$ are mapped to the same point of $\bar{Q}$ if and only if there is a sequence $(x_0, \ldots, x_n)$ such that $x = x_0$, $y = x_n$ and $x_{i+1} = A(F_i)(x_i)$, where $x_i \in F_i$ and $x_{i+1} \in R(F_i)$. (Here (R, A) is the glueing data.) We may take $n \leqslant N$. It follows easily from compactness and the finiteness of the situation that the map $X \to \bar{Q}$ is closed. Therefore $\bar{Q}$ is hausdorff. □

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