

## 5. The number of Hadamard matrices of order $n$

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and therefore

$$\alpha_\gamma = \frac{1}{2^{N-\alpha-\dim L} \alpha!} P_K^{(\alpha)}(-1).$$

Multiplying both sides by  $2^{n-\dim L}$ , and plugging in equation (1), we obtain the claimed formula for  $|f^{-1}(v)|$ .  $\square$

COROLLARY 5. *Let  $v_{min}$  be the least value assumed by  $f$  on binary points. Then*

$$\frac{1}{2} (N + v_{min}) = \text{the order of } -1 \text{ as a root of } P_K(T). \quad \square$$

### 5. THE NUMBER OF HADAMARD MATRICES OF ORDER $n$

A *Hadamard matrix* is a square matrix  $H$  of order  $n$  with entries in  $\{+1, -1\}$ , satisfying the relation

$$H \cdot H^T = nI_n.$$

( $H^T$  denotes the transpose of  $H$ , and  $I_n$  the identity matrix of order  $n$ .)

It is well known that the order of a Hadamard matrix can only be 1, 2 or a multiple of 4. Conversely, the existence of a Hadamard matrix of order  $n$  for every  $n \equiv 0 \pmod{4}$  is a longstanding conjecture, due to Jacques Hadamard [H]. The smallest open case currently occurs at  $n = 428$ . For a survey on Hadamard matrices, see [SY].

The theory exposed above yields a counting formula for Hadamard matrices of order  $n$ , in terms of the weight enumerator of a certain binary linear code of length  $\binom{n}{2}^2$ .

#### STEP 1. *Defining equations for Hadamard matrices.*

We represent binary matrices of order  $n$  as points  $p = (p_{i,j}) \in \{1, -1\}^{n^2}$ . Considering  $n^2$  variables  $\{x_{i,j}\}_{1 \leq i,j \leq n}$ , let

$$g_{k,l} = \sum_{r=1}^n x_{k,r} x_{l,r}.$$

If  $p = (p_{i,j})$  is a binary matrix, then  $g_{k,l}(p)$  is the dot product of the  $k$ -th and  $l$ -th rows of  $p$ . Thus, a binary matrix  $p$  is Hadamard if and only if

$$g_{k,l}(p) = 0 \quad \text{for all } 1 \leq k < l \leq n.$$

STEP 2. *Reduction to a single equation.*

Let

$$g = \sum_{1 \leq k < l \leq n} g_{k,l}^2.$$

By construction, we have the following properties:

- (1)  $g(p) \geq 0$  for every binary matrix  $p$ ;
- (2)  $g(p) = 0$  if and only if  $p$  is Hadamard.

Developing the expression for  $g$ , we obtain:

$$\begin{aligned} g &= \sum_{k < l} g_{k,l}^2 \\ &= \sum_{k < l} (\sum_r x_{k,r} x_{l,r})^2 \\ &= \sum_{k < l} (n + 2 \sum_{r < s} x_{k,r} x_{l,r} x_{k,s} x_{l,s}) \\ &= n \binom{n}{2} + 2f, \end{aligned}$$

where

$$f := \sum_{k < l} \sum_{r < s} x_{k,r} x_{l,r} x_{k,s} x_{l,s}.$$

(Of course, the above computation is performed modulo the relations  $x_{i,j}^2 = 1$  for all  $i, j$ .)

The following properties of  $f = \frac{1}{2} (g - n \binom{n}{2})$  derive instantly from those of  $g$ :

- (1)  $f(p) \geq -\frac{1}{2} n \binom{n}{2}$  for every binary matrix  $p$ ;
- (2)  $f(p) = -\frac{1}{2} n \binom{n}{2}$  if and only if  $p$  is Hadamard.

STEP 3. *The code associated with  $f$ .*

Let  $K_n := L_f^\perp$  denote the dual of the binary code  $L_f$  associated with  $f$ , as defined in Section 3. Explicitly, we consider the map

$$\begin{aligned} \phi_n: \quad \mathbf{F}_2^{\binom{n}{2}} &\rightarrow \mathbf{F}_2^{n^2} \\ E(k, l; r, s) &\mapsto e_{k,r} + e_{l,r} + e_{k,s} + e_{l,s}, \end{aligned}$$

where  $\{E(k, l; r, s)\}_{1 \leq k < l \leq n, 1 \leq r < s \leq n}$  and  $\{e_{i,j}\}_{1 \leq i, j \leq n}$  denote the standard bases of the left and right spaces, respectively; by construction then,  $K_n = \text{Ker}(\phi_n)$ .

As a direct consequence of Theorem 4 and of the above-mentioned properties of  $f$ , we obtain the

**THEOREM 6.** Let  $K_n$  ( $n$  even) be the code of length  $\binom{n}{2}^2$  defined as the kernel of the above map  $\phi_n: \mathbf{F}_2^{\binom{n}{2}^2} \rightarrow \mathbf{F}_2^{n^2}$ . Let  $P_n(T)$  denote the weight enumerator of  $K_n$ . Then the number  $h(n)$  of Hadamard matrices of order  $n$  is given by

$$h(n) = \frac{1}{2^{\beta(n)} \alpha(n)!} \cdot P_n^{(\alpha(n))}(-1),$$

where

1.  $\alpha(n) = n^2(n - 1)(n - 2)/8$ ;
2.  $\beta(n) = n^3(n - 1)/8 - n^2$ ;
3.  $P_n^{(\alpha(n))}(-1)$  denotes the  $\alpha(n)$ -th derivative of  $P_n(T)$ , evaluated at  $-1$ .

*Proof.* In the formula of Theorem 4, replace:

- $N$ , the length of the code, by  $\binom{n}{2}^2$ ;
- $\nu$ , a lower bound for the values of  $f$ , by  $-\frac{1}{2}n \binom{n}{2}$ ; and
- $n$ , the number of variables in  $f$ , by  $n^2$ . □

Thus, the determination of the weight enumerator of  $K_n$  is an important problem. We will give below, without proof, the number of codewords of weight 3, 4 and 5 of  $K_n$ . (Of course, there are no words of weight 1 or 2 in  $K_n$ .) But the problem can be generalized a little bit, as follows. Consider the map

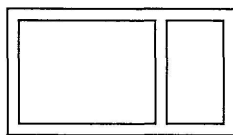
$$\begin{aligned} \phi_{m,n}: \mathbf{F}_2^{\binom{m}{2} \binom{n}{2}} &\rightarrow \mathbf{F}_2^{mn} \\ E(k, l; r, s) &\mapsto e_{k,r} + e_{l,r} + e_{k,s} + e_{l,s}, \end{aligned}$$

where now, the indices  $k < l$  range from 1 to  $m$  instead of 1 to  $n$ . We denote by  $K_{m,n}$  the kernel of  $\phi_{m,n}$ .

Let  $\Gamma = \{1, \dots, m\} \times \{1, \dots, n\}$ . We can think of the vector basis  $e_{i,j}$  as the point on row  $i$  and column  $j$  in the grid  $\Gamma$ , and of  $E(k, l; r, s)$  as the rectangle determined by rows  $k, l$  and columns  $r, s$  in  $\Gamma$ . The image of  $E(k, l; r, s)$  under  $\phi_{m,n}$ , then, is the formal sum of its four corners.

Thus, an element of weight  $w$  in  $K_{m,n}$  can be pictured as a set of  $w$  rectangles in the grid  $\Gamma$ , such that every point in the grid coincides with an *even* number of corners of the rectangles in the set.

For example, all elements of weight 3 in  $K_{m,n}$  can be represented (up to proper size and location) by the following picture:



or its vertical analogue. This picture represents a codeword of the form

$$E(k, l; r_1, r_2) + E(k, l; r_1, r_3) + E(k, l; r_2, r_3).$$

Thus, the number of codewords of weight 3 in  $K_{m,n}$  is equal to

$$w_3(K_{m,n}) = \binom{m}{2} \binom{n}{3} + \binom{m}{3} \binom{n}{2}.$$

Similarly, one can show that

$$w_4(K_{m,n}) = 3 \binom{m}{2} \binom{n}{4} + 9 \binom{m}{3} \binom{n}{3} + 3 \binom{m}{4} \binom{n}{2};$$

$$w_5(K_{m,n}) = 12 \binom{m}{2} \binom{n}{5} + 72 \binom{m}{3} \binom{n}{4} + 72 \binom{m}{4} \binom{n}{3} + 12 \binom{m}{2} \binom{n}{5} + 9 \binom{m}{3} \binom{n}{3}.$$

As a last remark, note that an upper bound for the weights in the associated code  $L_f$  is given by  $\frac{1}{8}n^3(n-1)$ , and that this bound is actually attained for some  $n$  if and only if there exists a Hadamard matrix of order  $n$ . This follows from, say, Corollary 3.

## 6. THE NUMBER OF PROPER 4-COLORINGS OF A GRAPH

Let  $G = (V, E)$  be a simple graph (no loops, no multiple edges) with vertex set  $V$  and edge set  $E$ . We will identify  $V$  with  $\{1, \dots, n\}$ , and denote the cardinality of  $E$  by  $e$ .

A 4-coloring of  $G$  is the assignment to every vertex of one among four fixed colors; such a coloring is *proper* if the colors assigned to the end vertices of any edge are distinct. For a survey on the 4-colorings of planar graphs, see [SK].

We will count the number of proper 4-colorings of  $G$ , in terms of the weight enumerator of a certain code of length  $3e$ .

STEP 1. *The defining equations for proper 4-colorings.*

As our palette of colors, we will choose the 4-set  $\{1, -1\}^2$ . The space of all 4-colorings of  $G$  can thus be identified with  $\{1, -1\}^{2n}$ , for example as follows: