

## 2. Background and Definitions

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## 2. BACKGROUND AND DEFINITIONS

Let  $X$  be a topological space and  $f$  a homeomorphism of  $X$ . We say that  $f$  is periodic if there is an integer  $n > 0$  such that  $f^n = Id$ . The period of  $f$  is the smallest positive integer  $n$  with this property.

As we will use them without further justifications, let us first recall some basic properties of one-dimensional maps.

Let  $f: I \rightarrow I$  be a periodic homeomorphism of the unit interval. If  $f$  preserves the endpoints then  $f$  is the identity map. If  $f$  exchanges the endpoints then  $f^2 = Id$  and  $f$  is conjugate to the reflection map  $x \mapsto 1 - x$ . Similarly, a periodic homeomorphism of the real line  $\mathbf{R}$  is the identity map or is a conjugate of the involution  $x \mapsto -x$  according to whether it is an increasing or a decreasing function.

Let  $f: S^1 \rightarrow S^1$  be a periodic homeomorphism of period  $n$  of the unit circle. If  $f$  is order-preserving then the rotation number of  $f$ ,  $\rho(f) = k/n$ , where  $k$  and  $n$  are coprime (see [5] for an excellent exposition on rotation numbers) and  $f$  is conjugate to a rotation of angle  $2k\pi/n$ . If  $f$  is order-reversing then  $f$  has exactly two fixed points,  $f^2$  is the identity map and the two arcs delimited on  $S^1$  by the fixed points of  $f$  are permuted by  $f$ .

A metric space  $X$  is path connected if there exists a continuous map from the unit interval  $[0, 1]$  into  $X$  which joins any two given points. It is arcwise connected if there is a topological embedding of  $[0, 1]$  into  $X$  which joins any two given distinct points. In fact, it can be shown that the two notions are equivalent (see [14, Theorem 4.1] or [11, Lemma 16.3]).

**LEMMA 2.1.** *A metric space  $X$  is path connected if and only if it is arcwise connected.*

A useful characterisation of path connected spaces is given in term of local connectivity. A metric space  $X$  is locally connected if each point of  $X$  possesses arbitrary small connected neighbourhoods. The following can be shown (see [8, Theorem 3.15] or [11, Lemma 16.4]):

**LEMMA 2.2.** *A compact, connected and locally connected metric space is pathwise connected.*

Another important ingredient used in this article, and in fact the ultimate result we will need, is the famous Jordan-Schoenflies theorem on simple closed curves in the plane (see [2, 9] or [12, Theorem 17.1]).

**THEOREM 2.3 (Jordan-Schoenflies).** *Every simple closed curve  $J$  divides the plane into exactly two components of each of which it is the*

complete boundary and the closure of the bounded component can be mapped topologically onto the closed unit disc.

In what follows, a *closed* topological disc (or just a topological disc)  $D$  is the image under a topological embedding of the *closed* unit disc and we write  $D^\circ$  for its interior and  $\partial D$  for its boundary. However, the closure of a bounded open set which is homeomorphic to the open unit disc is not necessarily a closed topological disc [11, Chapter 15].

**PROPOSITION 2.4.** *Let  $D_1, D_2, \dots, D_n$  be a finite number of closed topological discs in the plane and  $J^\circ$  be any connected component of  $\bigcap_{i=1}^n D_i^\circ$ . Then  $\partial J$  is a simple closed curve and  $J$  the closure of  $J^\circ$  is a topological disc.*

*Proof of 2.4.* We will use induction on  $n$ , the number of discs. If  $n = 1$  this is just the Jordan-Schoenflies theorem, so let us suppose that the result holds for some  $n (n \geq 1)$  and let  $J^\circ$  be any component of the complement of  $n + 1$  topological discs  $D_1, D_2, \dots, D_{n+1}$  in the plane. Let  $K^\circ$  be the component of  $\bigcap_{i=1}^n D_i^\circ$  that contains  $J^\circ$ . By induction, its closure  $K$  is a topological disc. Since  $J^\circ$  is a component of  $K^\circ \cap D_{n+1}^\circ$ , it suffices to show that the result holds for two discs  $D_1$  and  $D_2$  (see Figure 1). Set  $C_i = \partial D_i$  for  $i = 1, 2$  and let  $J$  be the closure of a component of  $D_1^\circ \cap D_2^\circ$ . We have that  $\partial J \neq \emptyset$  and  $\partial J \subset C_1 \cup C_2$ . If  $\partial J$  is entirely contained in one of the two curves, say  $C_1$ , then  $J = D_1$  and the lemma is proved. We can thus suppose that  $\partial J \not\subset C_1$  and  $\partial J \not\subset C_2$ .

Let  $x \in \partial J$ ,  $x \notin C_2$ . Then  $x \in C_1 \cap D_2^\circ$ , and we can find an arc  $\gamma$  in  $C_1$  such that:

$$x \in \gamma, \quad \gamma \subset \partial J, \quad \gamma \setminus \partial \gamma \subset D_2^\circ, \quad \partial \gamma \subset C_2.$$

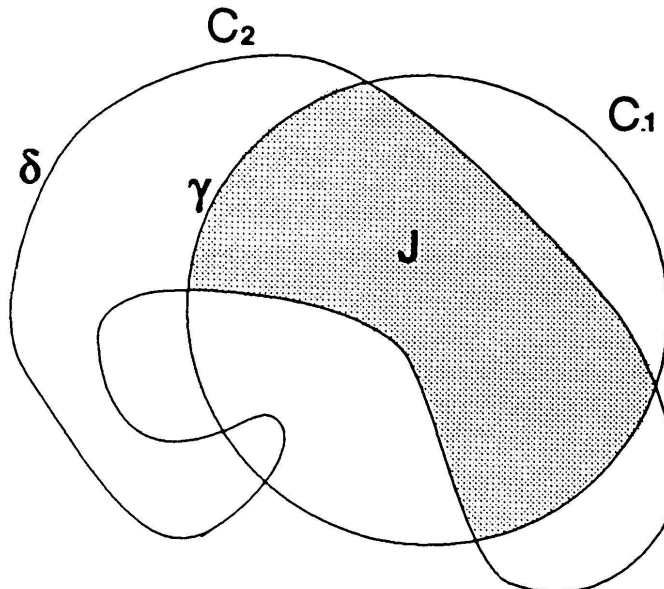


FIGURE 1

The endpoints of  $\gamma$  determine on  $C_2$  an arc  $\delta$  disjoint from  $J^o$  and such that  $\delta \cap J = \partial\delta$ . We note that there is an at most countable family of such arcs  $\gamma$ , noted  $(\gamma_i)_{i \in \mathbb{N}}$  and that  $\text{diam}(\gamma_i) \rightarrow 0$  as  $i \rightarrow \infty$ . The boundary of  $J$  is the simple closed curve obtained from  $C_2$  when substituting the arcs  $\gamma_i$  for the arcs  $\delta_i$  and  $J$  is a topological disc by the Jordan-Schoenflies theorem.  $\square$

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane  $\mathbf{R}^2$ , namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

**LEMMA 2.5.** *Let  $f: S \rightarrow S$  be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold  $S$  and let  $x \in \text{Fix}(f)$ , a fixed point of  $f$ . Then for any neighbourhood  $N$  of  $x$ , there exists a topological disc  $\Delta_x$  such that:*

1.  $\Delta_x \subset N$ ,
2.  $\Delta_x$  is a neighbourhood of  $x$ ,
3.  $f(\Delta_x) = \Delta_x$ .

*Proof of 2.5.* We can first assume that  $N$  and its image under  $f$ ,  $f(N)$ , are contained in some local chart  $U$  homeomorphic with  $\mathbf{R}^2$  and will continue to call  $x$  and  $N$  the corresponding point and set in  $\mathbf{R}^2$ . Let  $D_x$  be an euclidean disc of centre  $x$  and radius  $\eta$  where  $\eta > 0$  is chosen such that  $f^k(D_x) \subset N$  for  $k = 0, 1, \dots, n-1$  and let  $C_x$  be its boundary. Let  $\Delta_x$  be the closure of the component of the invariant set  $\bigcap_{k=0}^{n-1} f^k(D_x^o)$  which contains  $x$ . By 2.4,  $\Delta_x$  is a topological disc which is invariant under  $f$  (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma.  $\square$

*Remark.* The boundary  $\gamma_x$  of  $\Delta_x$ , which is an invariant simple closed curve, is contained in  $\bigcup_{k=0}^{n-1} f^k(C_x)$ .

### 3. PERIODIC HOMEOMORPHISMS OF THE DISC

**THEOREM 3.1.** *Let  $f: D^2 \rightarrow D^2$  be a periodic homeomorphism. Then there exists  $r \in O(2)$  and a homeomorphism  $h: D^2 \rightarrow D^2$  such that  $f = hrh^{-1}$ .*