3. Periodic Homeomorphisms of the Disc

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

The endpoints of γ determine on C_2 an arc δ disjoint from J^o and such that $\delta \cap J = \partial\delta$. We note that there is an at most countable family of such arcs γ, noted $(\gamma_i)_{i \in N}$ and that $diam(\gamma_i) \to 0$ as $i \to \infty$. The boundary of J is the simple closed curve obtained from C_2 when substituting the arcs γ_i for the arcs δ_i and J is a topological disc by the Jordan-Schoenflies theorem. □

The following remarkable property of periodic homeomorphisms which is a direct consequence of 2.4 is true in a more general setting than the plane $\mathbf{R}^2$, namely in topological manifolds of dimension 2 because of its local nature. We will give it in that context since we will use it for the disc and the sphere, repeatedly in this article.

LEMMA 2.5. *Let* $f : S \to S$ *be a periodic homeomorphism of an arbitrary 2-dimensional topological manifold* S *and let* $x \in Fix(f)$, *a fixed point of* f. *Then for any neighbourhood* N *of* x, *there exists a topological disc* Δ_x *such that:*

1. $\Delta_x \subset N$,

2. Δ_x *is a neighbourhood of* x,

3. $f(\Delta_x) = \Delta_x$.

Proof of 2.5. We can first assume that N and its image under f, $f(N)$, are contained in some local chart U homeomorphic with $\mathbf{R}^2$ and will continue to call x and N the corresponding point and set in $\mathbf{R}^2$. Let D_x be an euclidean disc of centre x and radius η where $\eta > 0$ is chosen such that $f^k(D_x) \subset N$ for $k = 0, 1, .., n-1$ and let C_x be its boundary. Let Δ_x be the closure of the component of the invariant set $\cap_{k=0}^{n-1} f^k(D_x^o)$ which contains x. By 2.4, Δ_x is a topological disc which is invariant under f (components are sent to components by a homeomorphism) and satisfies the three assertions of the lemma. □

Remark. The boundary γ_x of Δ_x, which is an invariant simple closed curve, is contained in $\cup_{k=0}^{n-1} f^k(C_x)$.

3. PERIODIC HOMEOMORPHISMS OF THE DISC

THEOREM 3.1. *Let* $f : D^2 \to D^2$ *be a periodic homeomorphism. Then there exists* $r \in O(2)$ *and a homeomorphism* $h : D^2 \to D^2$ *such that* $f = hrh^{-1}$.

Before attacking the proof of the result above, let us first look at a special case of Theorem 3.1, namely:

PROPOSITION 3.2. *Let* $f: D^2 \to D^2$ *be a periodic homeomorphism such that* $f/_{\partial D^2} = Id$. *Then* $f = Id$.

Proof of 3.2. Let d be an arbitrary diameter of D^2 with endpoints A and B and let Δ be one of the two connected components of $D^2 - d$. The set:

$$E = \bigcap_{i=1}^{n} f^i(\Delta^o)$$

is invariant under f and the closure of each of its components is a topological disc.

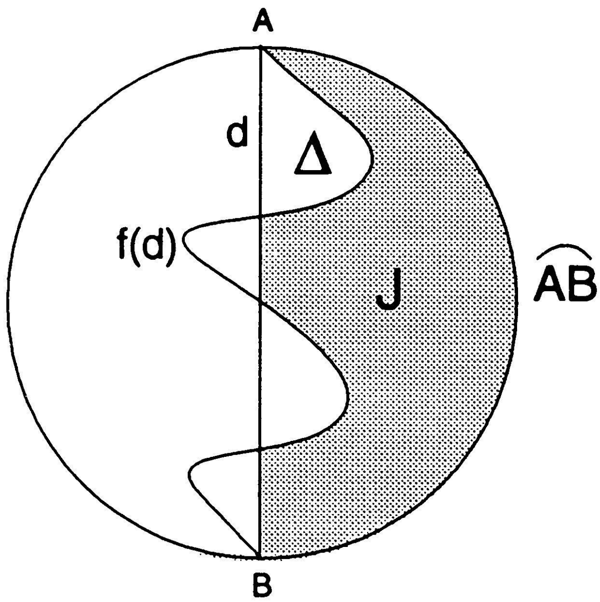

FIGURE 2

Let $\widehat{AB}$ be the arc of circle joining A to B in the boundary of Δ. Since $f^i(\widehat{AB}) = \widehat{AB}$ for all i, there exists a component of E, say J^o, whose closure J contains $\widehat{AB}$ (see Figure 2). By 2.4, J is a topological disc which is invariant under f.

We can write $\partial J = \widehat{AB} \cup \delta$ where δ is an f-invariant, simple arc with endpoints A and B such that:

$$\delta \subset \bigcup_{i=1}^{n} f^i(d) .$$

Since $f(A) = A$ and $f(B) = B$, $f/_\delta = Id$. Let x be a point of the arc δ. There exists $i \in \{1, \dots, n\}$ such that $x \in f^i(d)$ and $x = f^{n-i}(x) \in d$ so that $\delta = d$ and $f/_d = Id$. Since the diameter d was chosen arbitrarily, we have shown that $f = Id$ on D^2. □

From now on, f will denote a periodic homeomorphism of the disc of period n with $n > 1$. In the sequel of this section, we prove Theorem 3.1, first investigating the structure of the fixed point set of f.

PROPOSITION 3.3. *Suppose* $f : D^2 \mapsto D^2$ *is a periodic homeomorphism of period* n $(n > 1)$; *then:*

1. *if* f *is orientation-preserving,* $Fix(f)$ *is reduced to a single point which is not on the boundary of* D^2 *and for* $1 \leqslant i \leqslant n-1$, $Fix(f^i) = Fix(f)$;
2. *if* f *is orientation-reversing,* $f^2 = Id$ *and* $Fix(f)$ *is a simple arc which divides* D^2 *into two topological discs which are permuted by* f.

Proof of 3.3. Suppose first that f is orientation-preserving. By Brouwer fixed point theorem, f has at least one fixed point. Since $f/_{\partial D^2}$ is orientation-preserving and periodic, f has no fixed point on ∂D^2. Otherwise f would be the the identity map on ∂D^2 and using 3.2, f would be the identity map on the whole disc which is excluded by hypothesis. Therefore, f has at least one fixed point in $D^2 \backslash \partial D^2$ which we can assume to be, up to conjugacy, O, the center of the disc.

Let $A = D^2 \backslash \{O\}$. A is a half open annulus which is invariant under f. Suppose now that an iterate f^i of f has a fixed point $x_0 \in A$. Let $\tilde{x}_0$ be a lift of x_0 to the universal covering space $\tilde{A}$ of A and G be the lift of f^i such that $G(\tilde{x}_0) = \tilde{x}_0$. G^n is a lift of Id which fixes one point, thus $G^n = Id$. In particular, $G/_{\partial \tilde{A}}$ is a periodic and orientation preserving homeomorphism of the line, thus $G = Id$ on $\partial \tilde{A}$. Therefore, $f^i = Id$ on ∂D^2 and, according to 3.2, $f^i = Id$ on the whole disc, so that i is a multiple of n according to the definition of n.

Suppose now that f is orientation-reversing. In that case, f has exactly two fixed points on ∂D^2 which we denote by A and B and f^2 is the identity map on ∂D^2, therefore, by 3.2, $f^2 = Id$ on D^2 .

We assert that $Fix(f)$ is connected. For if not, we can find two nonempty compact sets K_1 and K_2 such that

$$Fix(f) = K_1 \cup K_2, \qquad K_1 \cap K_2 = \emptyset .$$

If $A \in K_1$ and $B \in K_2$, it is then possible to construct a simple arc γ in $D^2 \backslash (K_1 \cup K_2)$ which intersect ∂D^2 only on its endpoints and which separates A from B. Using the same argument as the one used in the proof of 3.2, we can show the existence of an f-invariant simple arc:

$$\delta \subset \bigcup_{i=0}^{n-1} f^i(\gamma) \subset D^2 \setminus Fix(f)$$

which separates A from B. But f must then have a fixed point on δ which gives a contradiction. Therefore we can suppose that one of the two compact sets, say K_1 is contained in $D^2 \setminus \partial D^2$. In that case, it is possible to construct a simple closed curve $c \subset D^2 \setminus \partial D^2$ which does not meet $K_1 \cup K_2$ and such that the topological disc it bounds contains at least one point of K_1. Using similar arguments as those of the proof of 2.5, we can find an f-invariant topological disc in $D^2 \setminus \partial D^2$ whose boundary contains no fixed point. This gives again a contradiction, since any simple closed curve which bounds an invariant disc has exactly two fixed points of f.

The previous arguments applied to an arbitrarily small invariant topological disc around a fixed point given by 2.5 shows that $Fix(f)$ is also locally connected and by 2.2, $Fix(f)$ is therefore pathwise connected. In view of 2.1, there exists a simple arc γ in $Fix(f)$ which joins A and B. This arc divides D^2 into two topological discs Δ_1 and Δ_2 by the Jordan-Schoenflies theorem. $D^2 \setminus \gamma$ is obviously invariant under f and the two arcs on ∂D^2 delimited by A and B are permuted by f, therefore $f(\Delta_1) = \Delta_2$, $f(\Delta_2) = \Delta_1$ and $Fix(f)$ is reduced to γ. □

Proof of 3.1. Suppose first that f is orientation-preserving. By 3.3, we can suppose that $Fix(f) = \{O\}$, the center of the disc. Since $f/_{\partial D^2}$ is a periodic homeomorphism of period n, the rotation number of $f/_{\partial D^2}$, $\rho(f/_{\partial D^2}) = k/n$, where k and n are coprime. We are going to prove that f is conjugate to a rotation by angle $2k\pi/n$ around the origin. Without loss of generality, we can assume that $k = 1$. Indeed, suppose the result holds if $\rho(f/_{\partial D^2}) = 1/n$. Then, if $k > 1$ we replace f by f^j where $j \in \mathbf{N}$ is such that $jk \equiv 1 (mod\, n)$. Then $\rho(f^j/_{\partial D^2}) = 1/n$, thus f^j is conjugate to a rotation by angle $2\pi/n$ around the origin and since $(f^j)^k = f$, it follows that f is conjugate to a rotation by angle $2k\pi/n$.

Let us consider the quotient space $D^2/_f$ where two points are identified if they belong to the same orbit under f. $D^2/_f$ is endowed with the quotient topology. It is a compact and pathwise connected metric space, the metric being defined by:

$$d(\pi(x), \pi(y)) = \inf_{0 \leqslant h, k \leqslant n-1} \{d(f^k(x), f^h(y))\},$$

where $\pi : D^2 \to D^2/_f$ is the canonical projection.

By 2.1, we can find a simple arc γ from $\pi(O)$ to an arbitrary point on $\pi(\partial D^2)$. Since the group of homeomorphisms generated by f acts freely on D^2 except at O it follows that $\pi : D^2 \to D^2/_f$ is a regular branched covering (see [10] page 49). Therefore, $\pi^{-1}(\gamma)$ is the union of n disjoint simple arcs (with the exception of their common endpoint O) $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$, which divide D^2 into n disjoint sectors, $A_0, A_1, \ldots, A_{n-1}$. The hypothesis $\rho(f/_{\partial D^2}) = 1/n$ implies that $\gamma_i = f^i(\gamma_0)$.

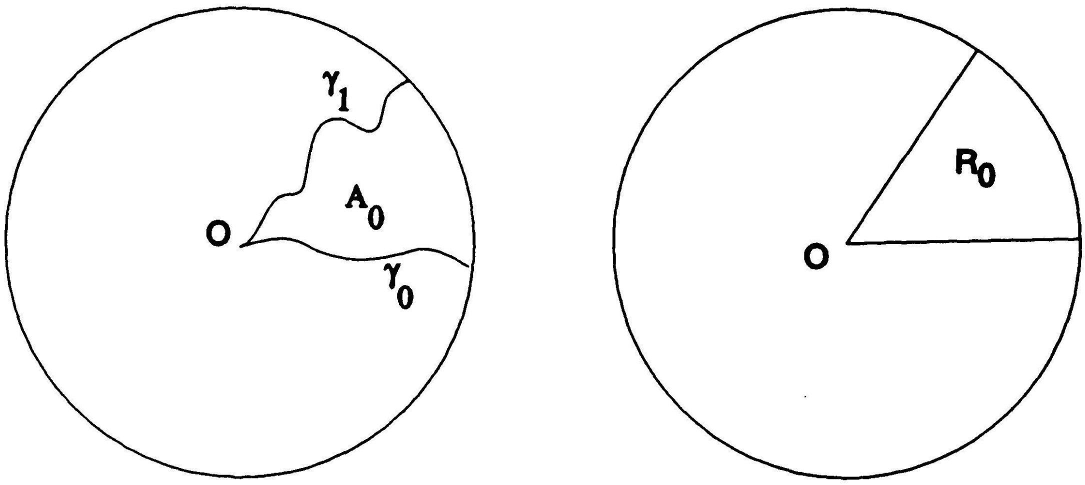

FIGURE 3

Let h be a homeomorphism between A_0 and R_0, the fundamental region in D^2 of the rotation by angle $2\pi/n$ around the origin, and such that $h\backslash_{\gamma_1} = rh\backslash_{\gamma_0}$. We can extend h to a homeomorphism of D^2 by defining $h/_{A_i}$ as $r^i h f^{-i}$, r being the rotation of centre O and angle $2\pi/n$. It is easy to verify that h is an homeomorphism of D^2 and that $f = h^{-1}rh$.

Suppose now that f is orientation-reversing. By 3.3, $Fix(f)$ is a simple arc γ which divides D^2 into two topological discs Δ_1 and Δ_2 which are permuted by f. Let h be a homeomorphism between Δ_1 and the upper half disc D_1. We define h on Δ_2 in the following way:

$$h(y) = Sh/_{\Delta_1} f(y), \; y \in \Delta_2 ,$$

where S is the reflection about the x-axis. It is then easy to verify that h is a homeomorphism of D^2 and this gives a conjugacy between f and S. □

Remark. Using 3.1, it can also be shown that any periodic homeomorphism of the annulus is topologically equivalent to an euclidean isometry (modulo a flip of the boundary if it is not boundary-preserving).