4. Periodic homeomorphisms of the sphere

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **40 (1994)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

4. Periodic homeomorphisms of the sphere

The main result of this section is

THEOREM 4.1. *Let* $f: S^2 \to S^2$ *be a periodic homeomorphism. Then there exists* $r \in O(3)$ *and a homeomorphism* $h: S^2 \to S^2$ *such that* $f = hrh^{-1}$.

Proof of 4.1. We will divide the proof of Theorem 4.1 into two cases according to whether or not f has at least one fixed point.

Suppose first that f has a fixed point. Using 2.5, we deduce the existence of an invariant simple closed curve c which divides S^2 into two invariant discs D_1 and D_2.

If f is orientation preserving and $f \neq Id$, then f has no fixed point on c (cf. 3.2). Therefore, by Brouwer's fixed point theorem we know then that f has at least two fixed points; after a conjugacy, we can suppose that f fixes the two poles N and S of S^2. Using the results of last section, we are able to find n arcs joining N and S such that their union is an invariant set under f. As in Section 3, we can then construct a conjugacy between f and a rotation by angle $2k\pi/n$ around the South-North axis.

If f is orientation-reversing, then f has two fixed points on c. In each of the invariant disc D^1 and D^2, the fixed point set of f consists of a simple arc which joins the two fixed points of f on c. The union of these two arcs is a simple closed curve which coincides with the fixed point set of f on S^2. It is then easy to construct a conjugacy between f and the reflection about the equator.

Let now suppose that f has no fixed point on S^2. Up to conjugacy, we can assume that the second iterate of f, f^2 is a periodic rotation around the North-South axis. In particular the points N and S are exchanged by f. For $t \in (-1, 1)$, let C_t be the circle obtained by cutting the sphere by the plane $z = t$, D_t the disc bordered by C_t on S^2 which contains N and:

$$t_0 = \inf\{t \in (-1, 1)\ ;\ \ D_t \cap f(D_t) = \emptyset\}\ .$$

We write $D = D_{t_0}$ and $C = C_{t_0}$ for convenience. Then D meets $f(D)$ on its boundary and only on its boundary (see Figure 4). Let $P_0 \in C \cap f(C)$ and $P_1, P_2, \ldots, P_{n-1}$, the orbit of P_0 under f. The points $P_0, P_2, \ldots, P_n$ and $P_1, P_3, \ldots, P_{n-1}$ are distinct because f^2 is a rotation of period $n/2$.

Suppose that there exists $i \in \{1, 3, \ldots, n-1\}$ such that P_0 and $P_i = f^i(P_0)$ coincide. Then P_0, S and N are fixed by f^{2i} so $f^{2i} = Id$. Therefore $2i = n$.

Let b_0 be the arc of great circle that joins N to P_0 in D and $b_{n/2}$ its image under $f^{n/2}$. Then $b = b_0 \cup b_{n/2}$ is a simple arc joining N and S and not meeting its first $(n/2) - 1$ iterates under f away from N and S. These arcs divide the sphere into $n/2$ sectors and we can build a conjugacy between f and the composition of a rotation of period $n/2$ around the North-South axis with a reflexion about the equator.

Suppose now that the points $P_0, P_1, \ldots, P_{n-1}$ are distinct. Let b_0 an arc of great circle joining N and P_0 in D and b_0' an arc joining S to P_0 in $f(D)$ disjoint from $f(b_0)$ and from its first $n-1$ iterates (which is possible since f^2 is a rotation). The union of these two arcs is again a simple arc joining N and S which does not meet its first $n-1$ iterates under f away from N and S. The union of this arc and its iterates divides the sphere S^2 into n disjoint sectors. In that case, f is topologically equivalent to the composition of a rotation of period n around the North-South axis with a reflexion about the equator. □

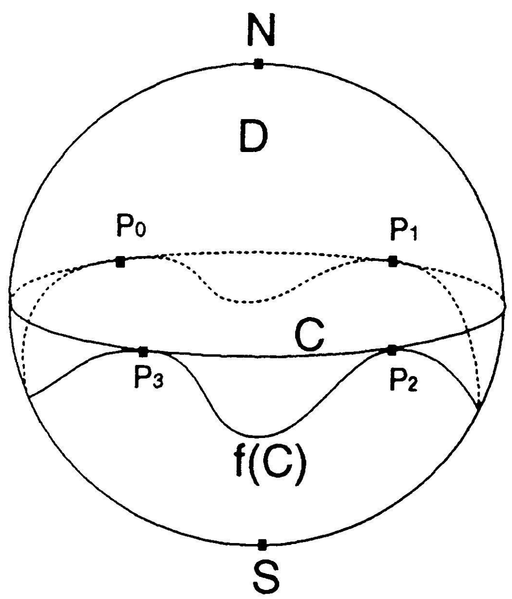

FIGURE 4

COROLLARY 4.2. *Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a periodic homeomorphism. Then f is topologically conjugate to a finite order rotation around the origin or to the reflexion about the x-axis.*

Proof of 4.2. We can extend f to a homeomorphism of the Sphere S^2 by identifying the plane $\mathbf{R}^2$ with the complement of the North pole using the stereographic projection. Looking at the proof of 4.1, f is either equivalent to a rotation around the North-South pole or to a reflexion about a great circle which we can assume to pass through the north pole N. It is not difficult to show that the conjugacy can be chosen to fix also the North pole N. This equivalence induces, therefore, a topological equivalence between f and a rotation or a reflexion about the x-axis. □

Remark. The investigation of periodic homeomorphisms on surfaces of positive genus has been studied extensively. We cannot give here a complete bibliography on the subject. We would just like to cite original works of Kerékjártó [4] and Nielsen [13] which lead to the conclusion that a periodic homeomorphism of a Riemannian surface of positive genus is conjugate to a conformal isometry.

Acknowledgement. The authors express their gratitude to Jerôme Fehrenbach, Lucien Guillou and Toby Hall for the discussions that helped to improve this paper.

REFERENCES

[1] BROUWER, L.E.J. Über die periodischen Transformationen der Kugel. *Math. Ann. 80* (1919), 39-41.

[2] CAIRNS, S.S. An elementary proof of the Jordan-Schoenflies theorem. *Proc. AMS 2* (1951), 860-867.

[3] KERÉKJÁRTÓ de, B. Über die periodischen Transformationen der Kreisscheibe und der Kugelfläche. *Math. Ann. 80* (1919-1920), 36-38.

[4] —— Über reguläre Abbildungen von Flächen auf sich. *Acta scient. math. Szeged 7* (1934), 65-75.

[5] DEVANEY, R. *An Introduction to Chaotic Dynamical Systems.* Benjamin-Commings 1986.

[6] EILENBERG, S. Sur les transformations périodiques de la surface de la sphère. *Fund. Math. 22* (1934), 28-44.

[7] EPSTEIN, D.B.A. Pointwise periodic homeomorphisms. *Proc. London Math. Soc. 42 (3)* (1981), 415-460.

[8] HOCKING, J.G. and G.S. YOUNG. *Topology.* Dover 1988.

[9] MAEHARA, R. The Jordan curve theorem via the Brouwer fixed point theorem. *Amer. Math. Monthly 91* (1984), 641-643.

[10] MASKIT, B. *Kleinian Groups.* A Series of Comprehensive Studies in Mathematics, Springer Verlag 1988.

[11] MILNOR, J. *Dynamics in one complex variable.* Preprint, Stony Brook, New York 1990.

[12] NEWMANN, M.H.A. *Elements of the Topology of Plane Sets of Points.* Cambridge Univ. Press (2nd edition) 1951.