PLURIDIMENSIONAL ABSOLUTE CONTINUITY FOR DIFFERENTIAL FORMS AND THE STOKES FORMULA

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INTRODUCTION

The concept of absolute continuity for functions of one real variable (defined on an open set $\Omega \subseteq \mathbf{R}$) arises very naturally in connection with the problem of characterizing the largest class of functions $u : \Omega \to \mathbf{R}$ for which there exists $f \in L^1(\Omega, \mathrm{loc})$ such that the Leibnitz-Newton formula

$$u(b) - u(a) = \int_a^b f(x)\,dx \tag{0.1}$$

holds for any interval $[a, b] \subseteq \Omega$. Lebesgue's solution to this problem, i.e. that (0.1) holds if and only if u is (locally) absolutely continuous, establishes the most general (and natural) framework within which the Fundamental Theorem of Calculus works.

Over the years, the subject has continuously received a great deal of attention. In particular, considerable effort in the literature was devoted to generalizing this result in various respects; see for instance the monographs [Wh], [Fe3], [Sa], [BM], [La], [Ju1], [Zi], and the references therein.

One of the early recognized directions was to try to allow less regular integrands by generalizing the Lebesgue integral. For instance, the existence of derivatives which are not Lebesgue integrable was regarded as a shortcoming of Lebesgue's integral and not as a pathology of the functions under discussion. This point of view eventually led to the design of the Denjoy-Perron integral in the 1910's (cf. e.g. [Sa], Chapters VI, VII). For more recent developments along these lines we refer to the work of Harrison [Ha], Henstock [H], Kurzweil [Ku], Pfeffer [P1, 2, 3, 4], Mawhin [M1, 2].

Nonetheless, there are other natural ways to extend Lebesgue's theorem to higher dimensions and to extend its validity to more general integrands and domains while still using the usual Lebesgue integral. See, for instance, Whitney [Wh], Bochner [Bo], Shapiro [Sh1, 2, 3] among others. Another very important and influential work but having somewhat different aims is that of Federer [Fe1, 2, 3].

There are two major aspects of the corresponding problem in the pluridimensional setting.

(i) The local problem (i.e. the validity aspect). Describe the class of $(n-1)$-forms u on a domain $\Omega \subseteq \mathbf{R}^n$ for which there exists a n-form $f \in L^1(\Omega, \text{loc})$ such that the following local Stokes formula holds:

$$\int_{\partial Q} u = \iint_Q f, \quad \text{for any rectangle } Q \subseteq \Omega\,. \tag{0.2}$$

(ii) The global problem (i.e. the invariant aspect). Find some minimal but also natural hypotheses on u so that the global Stokes formula

$$\int_{\partial\Omega} u = \iint_\Omega du \tag{0.3}$$

holds for a broad class of domains on C^1 manifolds.

The main goal of this work is to identify the essential analytical and geometrical assumptions needed to deal with *(i)* and *(ii)*. To treat the local problem we introduce the concept of absolute continuity for $(n-1)$-forms in $\mathbf{R}^n$. Being absolutely continuous turns out to be basically equivalent to the fact that the local Stokes formula holds true. It is important to point out that our definition is quite natural for it is homogeneous in n and reduces to the Lebesgue one when $n = 1$. Moreover, several alternative characterizations of this pluridimensional absolute continuity, much in the spirit of the one-dimensional results, can also be established.

Turning our attention to the global problem, let us first note that, due to the particular nature of the concept of rectifiability in the plane, the 2-dimensional case plays a special role in the literature. More concretely, many theorems initially stated in $\mathbf{R}^n$ can be further improved if $n = 2$ (see e.g. [P], [Lo], [JN]). However, since we shall try to formulate our main results with no artificial hypotheses and in as general a context as possible, we shall not attempt to single out this case in any way. Except for this particularity, our solution to the global problem is considerably more general than all the previously known forms of the Stokes theorem which go along the same coordinates. Moreover, both the validity context and its proof naturally reflect the scope of the theorem.

In addition to some necessary integrability assumptions, the differential form u satisfying (0.3) is assumed to be absolutely continuous and the singular set $S = (\bar{\Omega} \setminus \Omega) \cap \operatorname{supp} u$ is supposed to have $(n-1)$-dimensional Hausdorff measure zero, i.e. $\mu_{n-1}(S) = 0$. This should be compared, for instance, with Whitney's solution to the global problem in which the differential form u is assumed to be continuous and bounded outside of a singular set S satisfying certain geometric and measure theoretic conditions [Wh]. While these conditions do imply that $\mu_{n-1}(S) = 0$, the converse is, in general, false.

The key ingredient of the approach we present here is a localization method enabling us to pass from local, and even from infinitesimal, to global which we formalize and present in an axiomatic way. This is a synthesis as well as a significant extension of several basic procedures utilizing subdivision techniques. We refer to (the proofs of) Cousin's principle, Goursat's lemma, Pompeiu's removability theorem, etc.

The layout of the paper is as follows. The class of absolutely continuous differential forms is introduced and studied in § 1 and § 2. Among other things, here we show that for such forms the local Stokes formula is valid for arbitrary compact Lipschitz domains in place of rectangles. The localization technique alluded to earlier is devised in § 3. Global forms of the Stokes formula are obtained in § 4 for Lipschitz domains in $\mathbf{R}^n$ and, in invariant form, in § 5.

The last two sections are devoted to applications. The main results of § 6 give sufficient conditions under which the equalities $du = f$ and $\bar{\partial} u = f$ on $\Omega \setminus A$ (where A is a certain null set with a special structure) taken in the pointwise or in the distribution sense are actually valid on the entire domain Ω. In particular, for $f = 0$ and $u =$ function, we obtain very general removability criteria for holomorphic functions of several variables.

Finally, in §7, we record the Clifford algebra version of the results discussed in the previous sections: absolute continuity and the Leibnitz-Newton formula for Clifford-valued functions, removability criteria for monogenic functions, and the Pompeiu integral representation formula.

Before we begin the major part of this work, let us introduce some notation and definitions commonly used in the sequel. A *rectangle* in $\mathbf{R}^n$ will be any simplex Q of the form $Q := \prod_{k=1}^{n} [a_k, b_k]$, where $a_k, b_k \in \mathbf{R}$, $a_k < b_k$ for all k. The *eccentricity* of Q is given by

$$p(Q) := \sup_{1 \leqslant i, j \leqslant n} \frac{b_i - a_i}{b_j - a_j}.$$

Note that $p(Q) \geqslant 1$ and that Q is a *cube* precisely for $p(Q) = 1$. The lower left-most corner of Q, $(a_1, \ldots, a_n) \in \mathbf{R}^n$, will be called the *origin* of Q, whereas the upper right-most corner of Q, $(b_1, \ldots, b_n) \in \mathbf{R}^n$, the *end-point* of Q. The traces of the hyper-planes $\{x; x_k = a_k\}$ and $\{x; x_k = b_k\}$ on Q will be called the *faces* of Q. The collections of all rectangles contained in a subset Ω of $\mathbf{R}^n$ will be denoted by $\mathscr{R}(\Omega)$.

A *subdivision* of a rectangle Q is a finite collection of rectangles $(Q_i)_{i \in I}$ having mutually disjoint interiors and such that $\cup_{i \in I} Q_i = Q$. A subdivision of Q will be called *elementary* if its elements can be obtained as the Cartesian product of some fixed subdivisions of the factor intervals of Q.

More generally, the union $P = \cup_{i \in I} Q_i$ of finitely many rectangles $(Q_i)_{i \in I}$ with mutually disjoint interiors is called a *(compact) paved set,* and $(Q_i)_{i \in I}$ is said to be a *subdivision* of the paved set P.

The Euclidean space $\mathbf{R}^n$ is equipped with the usual metric $\|x\|^2 = \langle x, x \rangle = \sum_{i=1}^{n} x_i^2$, if $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. For $S \subset \mathbf{R}^n$, we set $\operatorname{diam}(S) := \sup\{\|x - y\|; x, y \in S\}$ and $\partial S := \bar{S} \setminus \overset{\circ}{S}$. Also, $\operatorname{comp}(S)$ will stand for the collection of all compact subsets of S. For $0 \leqslant r \leqslant n$, μ_r will denote the r-dimensional Hausdorff measure in $\mathbf{R}^n$, while λ_n will stand for the usual Lebesgue measure in $\mathbf{R}^n$. Finally, the $(n-1)$-dimensional and the n-dimensional Lebesgue integrals will be denoted by $\int$ and $\iint$, respectively.

1. Integral and absolute continuity and the local problem

Definition 1.1. *A bounded subset* Ω *of* $\mathbf{R}^n$ *is called a Lipschitz domain if for any* $a \in \Omega \setminus \mathring{\Omega}$, *there exists an open neighborhood* U *of* a *in* $\mathbf{R}^n$, *a coordinate system (isometric to the canonical one)* $(x', x_n) = ((x_1, \ldots, x_{n-1}), x_n)$, *and a Lipschitz continuous function* $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ *such that*

$$\Omega \cap U = \{(x', x_n); \varphi(x') \leqslant x_n\} \cap U .$$

Also, if the new coordinates are actually obtained by permuting the canonical ones, then Ω *is called a simple Lipschitz domain.*

Note that, the *border* of the domain Ω, $b\Omega := \Omega \setminus \mathring{\Omega}$, is either the empty set or a $(n-1)$-dimensional Lipschitz submanifold of $\mathbf{R}^n$ (assumed with the standard induced orientation).

Let now Ω be a Lipschitz domain in $\mathbf{R}^n$ and ω an open set in $\mathbf{R}^{n-1}$. A locally bi-Lipschitz mapping $\varphi : \omega \to \Omega$ is called *Lipschitz embedding* provided φ maps ω homeomorphically onto $\varphi(\omega)$. Furthermore, if S is a topological space, $h : S \times \omega \to \Omega$ is called a *continuous family of Lipschitz embeddings* if $h_s := h(s, \cdot)$ is a Lipschitz embedding for each fixed $s \in S$, and if the mappings

$$S \ni s \mapsto \frac{\partial h_s}{\partial x_i} \in L^\infty(\omega, \mathrm{loc}), \quad i = 1, \ldots, n-1 , \tag{1.1}$$

are continuous. Here $L^\infty(\omega,\mathrm{loc})$ is endowed with the usual (Fréchet) topology given by uniform convergence on compact subsets of ω. Throughout this paper S will actually always be a locally closed subspace of some $\mathbf{R}^k$.

Let $L^1(\Omega, \mathrm{loc})$ stand for the vector space of differential forms with locally integrable coefficients on Ω. We consider this space endowed with the usual (locally convex) topology.

Definition 1.2. *A complex-valued* $(n-1)$*-form* u *defined on* Ω *is called integrally continuous if:*

(1) the form u *is locally* $(n-1)$*-integrable, i.e.* $\varphi^* u$ *is locally integrable on* ω *for any Lipschitz embedding* $\varphi : \omega \to \Omega$;

(2) the mapping $S \ni s \mapsto h_s^* u \in L^1(\omega, \mathrm{loc})$ *is continuous, for any continuous family of Lipschitz embeddings* $h : S \times \omega \to \Omega$.

Examples. Let $u = \sum_{i=1}^n (-1)^{i-1} u_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ be a $(n-1)$-form on Ω where, as usual, the symbol under the "hat" is omitted in the product.

(1) If for $1 \leqslant i \leqslant n$ the functions $u_i|_C$ are μ_{n-1}-integrable for any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$, then u is locally $(n-1)$-integrable. In particular, this is the case if u_i are locally bounded.

(2) Suppose that there exists a set $A \subset \Omega$ of zero $(n-1)$-dimensional Hausdorff measure such that $u_i|_{(\Omega \setminus A)}$ is continuous (in the induced topology) and $u_i|_{C \cap (\Omega \setminus A)}$ is μ_{n-1} integrable for any $1 \leqslant i \leqslant n$ and any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$. Then u is integrally continuous as well.

(3) For $n = 1$, a $(n-1)$-form u is a function and, in this case, the form u is integrally continuous if and only if the function u is continuous.

Recall the usual exterior derivative operator d. The main result of this section is the following.

THEOREM 1.3. *Consider a Lipschitz domain* Ω *in* $\mathbf{R}^n$. *Let* u *be an integrally continuous* $(n-1)$*-form on* Ω *and let* f *be a locally integrable n-form on* Ω. *The following are equivalent.*

(1) For any compact Lipschitz domain $K \subseteq \Omega$ *we have* $\int_{\partial K} u = \iint_K f$.

(2) For any rectangle $Q \in \mathscr{R}(\Omega)$ *we have* $\int_{\partial Q} u = \iint_Q f$.

(3) $du = f$ *in the distribution sense on* $\mathring{\Omega}$.

Before we proceed with the proof of this theorem, we shall prove a lemma. To state it, we need some more notation. Let χ be a positive, smooth, function supported in the closed unit ball in $\mathbf{R}^n$ and normalized such that $\iint_{\mathbf{R}^n} \chi\, dx = 1$. For $\varepsilon > 0$, set $\Omega_\varepsilon := \{x \in \mathbf{R}^n; \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and, for any $\Phi \in L^1(\mathring{\Omega}, \mathrm{loc})$, set

$$\Phi_\varepsilon(x) := \iint_{\mathbf{R}^n} \Phi(x - \varepsilon y)\, \chi(y)\, dy, \quad x \in \Omega_\varepsilon .$$

It is a well-known fact that $\Phi_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and that $\Phi_\varepsilon \to \Phi$ in $L^1(\mathring{\Omega}, \mathrm{loc})$ as ε tends to zero. For a locally integrable form u on Ω, u_ε is defined componentwise.

LEMMA 1.4. *Let* Ω *be a Lipschitz domain in* $\mathbf{R}^n$ *and let* u *be an integrally continuous* $(n-1)$*-form on* Ω. *Then:*

(1) $u \in L^1(\Omega, \mathrm{loc})$;

(2) $\varphi^* u_\varepsilon \to \varphi^* u$ *in* $L^1(\omega, \mathrm{loc})$ *as* ε *approaches zero, for any Lipschitz embedding* $\varphi : \omega \to \mathring{\Omega}$.

Proof. For each sufficiently small $\varepsilon > 0$, fixed for the moment, consider the continuous family of Lipschitz embeddings

$$h : \Omega_\varepsilon \times \omega_\varepsilon \ni (x, t) \mapsto (x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n) ,$$

where Ω_ε, defined above, stands for the space of parameters and ω_ε stands for a suitably small, open neighborhood of the cube $[-\varepsilon, \varepsilon]^{n-1}$. Obviously, $(h_x^* u)(t) = \Phi(x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \cdots \wedge dt_{n-1}$, for some function Φ. Since u is integrally continuous, the function

$$\Phi^\varepsilon(x) := \varepsilon^{1-n} \int_{[-\varepsilon,\varepsilon]^{n-1}} \Phi(x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \cdots \wedge dt_{n-1}$$

is continuous on Ω_ε. For any small, fixed x_n, the Lebesgue differentiation theorem yields that $\Phi^\varepsilon(\cdot, x_n) \to \Phi(\cdot, x_n)$, as $\varepsilon \to 0$, almost everywhere with respect to the $(n-1)$-dimensional Lebesgue measure on $\{x' \in \mathbf{R}^{n-1}; (x', x_n) \in \mathring{\Omega}\}$. Using Fubini's theorem we infer that $\Phi^\varepsilon \to \Phi$, as $\varepsilon \to 0$, almost everywhere on Ω. Thus, Φ is λ_n-measurable.

Next, let $Q = Q' \times Q_n$ be a rectangle in $\mathbf{R}^{n-1} \times \mathbf{R}$ which is contained in $\mathring{\Omega}$, and consider the continuous family of Lipschitz embeddings

$$k : Q_n \times Q' \ni (x_n, x') \mapsto (x', x_n) \in \Omega .$$

Hence, $k_{x_n}^* u = \Phi(\cdot, x_n) dx_1 \wedge \cdots \wedge dx_{n-1}$. As u is integrally continuous, the mapping

$$Q_n \ni x_n \mapsto \int_{Q'} |\Phi(x', x_n)| \, dx_1 \wedge \cdots \wedge dx_{n-1}$$

is continuous. In particular, the iterated integral

$$\int_{Q_n} \int_{Q'} |\Phi(x', x_n)| \, dx' \wedge dx_n$$

is finite. By Fubini's theorem, it follows that Φ is integrable on Q.

Now, if $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ the above reasoning gives that $u_1 = \Phi$ is integrable on Q. Likewise, $u_2, \dots, u_n$ are integrable on Q, and u is thus locally integrable on $\mathring{\Omega}$.

To conclude the proof of (1) it suffices to show that any $a \in \Omega \setminus \mathring{\Omega}$ has a compact neighborhood K in $\mathbf{R}^n$ such that u is integrable on $K \cap \Omega$. To see this, there is no loss of generality assuming that K is so that

$$K \cap \Omega = \{(x', x_n); x' \in Q', \varphi(x') \leqslant x_n \leqslant \varphi(x') + \varepsilon\} ,$$

where Q' is a rectangle in $\mathbf{R}^{n-1}$, $\varphi:\mathbf{R}^{n-1}\to\mathbf{R}$ is a Lipschitz function, and $\varepsilon>0$ is some fixed, sufficiently small number. This time we take the continuous family of Lipschitz embeddings

$$h':[0,\varepsilon]\times Q'\ni(s,x')\mapsto(x',\varphi(x')+s)\in\Omega$$

and proceed as before. Hence, (1) follows.

To see (2), let $u=\sum_{i=1}^{n}(-1)^{i-1}u_i\,dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n$ in Ω, so that we have $\varphi^*u=(\sum_i(u_i\circ\varphi)\Phi_i)\,dt_1\wedge\cdots\wedge dt_{n-1}$, where Φ_i are measurable functions, locally (essentially) bounded on ω. Similarly, $\varphi^*u_\varepsilon=(\sum_i((u_i)_\varepsilon\circ\varphi)\Phi_i)\,dt_1\wedge\cdots\wedge dt_{n-1}$.

Given a compact subset C of ω, we consider the continuous family of Lipschitz embeddings $(s,t)\mapsto\varphi(t)-s$, where t lies in an open neighborhood of C and s lies in a small open ball centered at the origin of $\mathbf{R}^n$. Let also $\theta>0$ be an arbitrary, fixed number. By the integral continuity of u, there exists $\delta>0$ such that

$$\int_C\left|\sum_{i=1}^{n}u_i(\varphi(t)-\varepsilon y)\Phi_i(t)-\sum_{i=1}^{n}u_i(\varphi(t))\Phi_i(t)\right|\chi(y)\,dt<\theta \tag{1.2}$$

for all $|y|\leqslant 1$ and $0<\varepsilon<\delta$. Since, by (1), the functions u_i are locally integrable on $\mathring{\Omega}$, the function

$$\varepsilon^{-n}\sum_i u_i(y)\Phi_i(t)\chi\left(\frac{\varphi(t)-y}{\varepsilon}\right),\quad \varepsilon>0\,,$$

is locally integrable on $\omega\times\mathring{\Omega}$. Integrating (1.2) against dy over the closed unit ball in $\mathbf{R}^n$ and then changing the order of integration, we obtain

$$\int_C|\varphi^*u_\varepsilon-\varphi^*u|\,d\mu_{n-1}\leqslant c_n\theta\,,$$

for some $c_n>0$ depending only on n. Since $\theta>0$ was arbitrary, the proof of the lemma is therefore complete. □

Proof of Theorem 1.3. Obviously (1) implies (2). Next, assume that (2) holds and let Q be an arbitrary rectangle in $\mathbf{R}^n$ contained in $\mathring{\Omega}$. It is then straightforward to see that, for a sufficiently small $\varepsilon>0$,

$$\int_{\partial Q}u_\varepsilon=\iint_Q f_\varepsilon\,.$$

Since u_ε is smooth, the standard form of Stokes formula gives that $\iint_Q du_\varepsilon = \iint_Q f_\varepsilon$. As Q was arbitrarily chosen, we see that $du_\varepsilon = f_\varepsilon$ on $\mathring{\Omega}_\varepsilon$ and, hence, by letting ε go to zero, $du = f$ in the distribution sense on $\mathring{\Omega}$. Thus, (2) $\Rightarrow$ (3).

Finally, we consider the implication (3) $\Rightarrow$ (1). Using a smooth partition of unity, it is not difficult to see that matters can be reduced to verifying (1.1) in the following cases:

(i) the support of u is included in the interior of K;

(ii) the domain K has the form

$$\{(x', x_n) \in [0, 1]^{n-1} \times [0, 1]; x_n \leqslant \varphi(x')\}\,, \tag{1.3}$$

for some Lipschitz function $\varphi : \mathbf{R}^{n-1} \to (0, 1)$.

We present the proof in the second case, as the proof the first case goes along the same lines and is somewhat simpler. Let us first note that, if K_ε is as in (1.3) except that φ has been replaced by $\varepsilon\varphi$, with $0 < \varepsilon < 1$, on account of the integral continuity of u we have

$$\int_{\partial K} u = \lim_{\varepsilon \to 1} \int_{\partial K_\varepsilon} u\,.$$

Hence, it suffices to prove the statement with K_ε in place of K or, in other words, assuming that the compact domain K from (1.2) is actually contained in $\mathring{\Omega}$. Furthermore, since by (1) $du_\varepsilon = f_\varepsilon$ on Ω_ε for all $\varepsilon > 0$, and since

$$\int_{\partial K} u_\varepsilon \to \int_{\partial K} u, \quad \iint_K f_\varepsilon \to \iint_K f$$

as $\varepsilon \to 0$ (the first convergence utilizes the integral continuity of u), there is no loss in generality if we assume that u and f are smooth forms in a neighborhood of K.

Consider now the bi-Lipschitz homeomorphism

$$h : [0, 1]^n \ni (x', x_n) \mapsto (x', x_n\varphi(x')) \in K\,.$$

From the change of variable formula ([Fe3], Theorem 3.2.3, p. 243) we have

$$\int_{\partial K} u = \int_{\partial[0,1]^n} h^* u\,.$$

Also, a routine calculation shows that

$$(h^*u)\,(x', x_n) = v(x', x_n) + \left(\sum_{i=1}^{n-1} w_i(x', x_n)\partial_i\varphi(x')\right) dx_1 \wedge \cdots \wedge dx_{n-1}\,,$$

where the coefficients of the $(n-1)$-form v as well as $(w_i)_i$ are Lipschitz functions. Clearly, the usual Stokes formula on $[0,1]^n$ holds for v whereas, for $1 \leqslant i \leqslant n-1$,

$$\int_{[0,1]^{n-1}} (w_i(x',1) - w_i(x',0))\partial_i\varphi(x')\,dx'$$

$$= (-1)^{n-1}\int_0^1 \int_{[0,1]^{n-1}} \frac{\partial w_i(x',x_n)}{\partial x_n}\,\partial_i\varphi(x')\,dx' \wedge dx_n\,.$$

Consequently, the Stokes formula holds for h^*u on $[0,1]^n$, so that

$$\int_{\partial K} u = \int_{\partial[0,1]^n} h^*u = \iint_{[0,1]^n} d(h^*u) = \iint_{[0,1]^n} h^*(du)$$

$$= \iint_{[0,1]^n} h^*f = \iint_K f$$

and the proof is complete. □

DEFINITION 1.5. *Let* Ω *be a Lipschitz domain in* $\mathbf{R}^n$. *An integrally continuous* $(n-1)$-*form* u *on* Ω *is called absolutely continuous on* Ω *if* $d(u\,|\,\mathring{\Omega})$, *taken in the distribution sense, is integrable on* $\mathring{K}$ *for any compact subset* K *of* Ω.

Note that if $u = \sum_{i=1}^n (-1)^{i-1} u_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and u_i are, for instance, locally Lipschitz on Ω, then u is absolutely continuous on Ω.

A simple consequence of Theorem 1.3 and of the above definition is the next.

THEOREM 1.6. *If* K *is a compact Lipschitz domain in* $\mathbf{R}^n$ *and* u *is an absolutely continuous* $(n-1)$-*form on* K, *then*

$$\int_{\partial K} u = \iint_{\mathring{K}} du\,.$$

2. CHARACTERIZATIONS OF THE PLURIDIMENSIONAL ABSOLUTE CONTINUITY

Theorem 1.3 suggests the possibility of characterizing pluridimensional absolute continuity of $(n-1)$-forms in a way similar to Lebesgue's definition of absolute continuity of functions on the real line (i.e. without involving the exterior derivative operator). This is made precise in the following theorem.

THEOREM 2.1. *Let* Ω *be an open subset of* $\mathbf{R}^n$ *and let* u *be a* $(n-1)$*-form which is locally* $(n-1)$*-integrable on* Ω. *The following are equivalent.*

(1) There exists a locally integrable n-form f *on* Ω *such that* $du = f$ *in the distribution sense on* Ω.

(2) There exists a locally integrable n-form f *on* Ω *such that* $\int_{\partial Q} u = \iint_Q f$, *for any* $Q \in \mathscr{R}(\Omega)$.

(3) There exists a locally integrable n-form g *on* Ω *such that* $|\int_{\partial Q} u| \leqslant \iint_Q g$, *for any* $Q \in \mathscr{R}(\Omega)$.

(4) For any $Q \in \mathscr{R}(\Omega)$ *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\sum_{i \in J} \left| \int_{\partial Q_i} u \right| \leqslant \varepsilon ,$$

for any subdivision $(Q_i)_{i \in I}$ *of* Q *and any* $J \subseteq I$ *for which* $\sum_{i \in J} \lambda_n(Q_i) \leqslant \delta$.

In particular, Theorem 1.3 and the above result show that an integrally continuous $(n-1)$-form u on Ω is absolutely continuous on Ω if it satisfies one of the above equivalent conditions. However, let us note that, without the integral continuity condition, u with (1)-(4) above is not necessarily absolutely continuous, except for $n = 1$.

Here is a simple counterexample in $\mathbf{R}^2$. If χ is the characteristic function of $\{(1-t, t); 0 \leqslant t \leqslant 1\} \subset \mathbf{R}^2$, then $u := \chi \, dx_2$ is locally 1-integrable and satisfies (1) – (4) in the above theorem, without being absolutely continuous on $\Omega := \mathbf{R}^2$.

Proof of Theorem 2.1. Clearly, all we need to show is that (4) implies (1). For each rectangle Q contained in Ω we set

$$\rho(Q) := \sup \left\{ \sum_{i \in I} \left| \int_{\partial Q_i} u \right| ; (Q_i)_{i \in I} \text{ an elementary subdivision of } Q \right\}. \tag{2.1}$$

Note that $|\int_{\partial Q} u| \leqslant \rho(Q) < +\infty$ for any rectangle Q. Also, since $Q \mapsto \int_{\partial Q} u$ is *rectangle-additive,* i.e. $\int_{\partial Q} u = \sum_{i \in I} \int_{\partial Q_i} u$ for any rectangle Q and any subdivision $(Q_i)_{i \in I}$ of Q, so is ρ. Therefore, it makes sense to extend ρ by setting

$$\rho(P) := \sum_{i \in I} \rho(Q_i) , \tag{2.2}$$

for any paved set P contained in Ω and any subdivision $(Q_i)_{i \in I}$ of P. The rectangle-additivity of ρ ensures that this extension is consistent with (2.1) and that (2.2) is independent of the particular choice of the subdivision $(Q_i)_{i \in I}$ of P. Going further, we extend ρ to $\mathrm{comp}(\Omega)$ by setting

$$\rho(K) := \inf\{\rho(P);\ P \text{ paved set, } K \subseteq P\}, \quad K \in \mathrm{comp}(\Omega) .$$

By (4), this extension is continuous in the sense that $\rho(K_\nu) \to \rho(K)$ whenever $(K_\nu)_\nu$ is a nested sequence of compact sets in Ω such that $\cap_\nu K_\nu = K$.

Now, for each multi-index $\alpha \in \mathbf{N}^n$ and for each $k \in \mathbf{N}$ we consider the cube $Q_{k,\alpha} := [0, 2^{-k}]^n + 2^{-k}\alpha$, and the set of multi-indices $I_k := \{\alpha \in \mathbf{N}^n;\ Q_{k,\alpha} \subseteq \Omega\}$. Moreover, for any complex-valued, continuous and compactly supported function ψ on Ω, we set

$$I_k(\psi) := \{\alpha \in I_k;\ \mathrm{supp}\,\psi \cap Q_{k,\alpha} \neq \varnothing\}$$

and

$$P_k(\psi) := \bigcup_{\alpha \in I_k(\psi)} Q_{k,\alpha} .$$

It follows that $P_{k+1}(\psi) \subseteq P_k(\psi)$ for any $k \in \mathbf{N}$ and that $\cap_k P_k(\psi) = \mathrm{supp}\,\psi$.

Next, we define

$$s_k(\psi) := \sum_{\alpha \in I_k(\psi)} \psi(2^{-k}\alpha) \int_{\partial Q_{k,\alpha}} u .$$

Clearly, s_k is a $\mathbf{C}$-linear functional on $C_0(\Omega)$ which satisfies

$$|s_k(\psi)| \leqslant \rho(P_k(\psi)) \sup_\Omega |\psi|, \quad \psi \in C_0(\Omega) .$$

Finally, we introduce $\mu : C_0(\Omega) \to \mathbf{C}$ by setting

$$\mu(\psi) := \lim_k s_k(\psi), \quad \psi \in C_0(\Omega) ,$$

where the existence of the limit easily follows from the uniform continuity of ψ. As μ is $\mathbf{C}$-linear and satisfies $|\mu(\psi)| \leqslant \rho(\mathrm{supp}\,\psi)\, \|\psi\|_{L^\infty}$, we infer that μ is a complex-valued Radon measure on Ω.

Fix $Q \in \mathscr{R}(\Omega)$ and take $\psi_\nu \in C_0(\Omega)$ a sequence of real-valued functions such that $0 \leqslant \psi_\nu \leqslant 1$ on Ω, $\psi_\nu = 1$ on a neighborhood of Q, $\mathrm{supp}\,\psi_{\nu+1} \subseteq \mathrm{supp}\,\psi_\nu$ and $\cap_\nu \mathrm{supp}\,\psi_\nu = Q$. From the definition of μ it is not difficult to see that

$$\left| \mu(\psi_\nu) - \int_{\partial Q} u \right| \leqslant \rho(\mathrm{supp}\,\psi_\nu) - \rho(Q) .$$

Hence, on account of the continuity of ρ, we see that $\int_Q d\mu = \int_{\partial Q} u$, for any Q. With this at hand and by once again using the hypothesis (4), we conclude that μ is absolutely continuous with respect to the n-dimensional Lebesgue measure λ_n. Therefore, if $f \in L^1(\Omega, \mathrm{loc})$ denotes the Radon-Nikodym-Lebesgue density of μ with respect to λ_n, we have that

$$\int_{\partial Q} u = \int_Q d\mu = \iint_Q f$$

for any $Q \in \mathscr{R}(\Omega)$. Using this, Theorem 1.3 finally implies that $du = f$ in the distribution sense on Ω, and this concludes the proof of the theorem. □

REMARK 2.2. Inspection of the proof also shows that $\rho(K) = \int_K |f| \, d\lambda_n$ for any compact subset K of Ω, and that $|f| \leqslant g$ a.e. on Ω.

An *integrally Lipschitz* $(n-1)$*-form* in Ω is a locally $(n-1)$-integrable form u for which there exists $M > 0$ so that

$$\left| \int_{\partial Q} u \right| \leqslant M \lambda_n(Q)$$

for each $Q \in \mathscr{R}(\Omega)$. Note that any integrally Lipschitz $(n-1)$-form u in Ω satisfies the equivalent conditions in Theorem 1.3.

3. A MAXIMUM PRINCIPLE

The purpose of this section is to prove a localization theorem which plays a significant role in the sequel. Here, our approach is of an abstract nature.

Let $\mathscr{X}$ be a fixed metric space. In general, for an arbitrary set E, we shall denote by $\mathscr{F}(E)$ the collection of all finite families of subsets of E, and by $\mathscr{S}(E)$ the collection of all subsets of $\mathscr{F}(E)$.

DEFINITION 3.1. *A rectangular system on* $\mathscr{X}$ *is a subset* $\mathscr{R}$ *of* $\mathrm{comp}(\mathscr{X})$ *together with an application* $\mathrm{div}: \mathscr{R} \to \mathscr{S}(\mathscr{R})$ *satisfying the following:*

(1) If $Q \in \mathscr{R}$ *and* $(Q_i)_{i \in I} \in \mathrm{div}(Q)$, *then* $Q_i \subseteq Q$ *for any* $i \in I$;

(2) For any $Q \in \mathscr{R}$ *and any* $\varepsilon > 0$, *there exists* $(Q_i)_{i \in I} \in \mathrm{div}(Q)$ *so that* $\mathrm{diam}(Q_i) < \varepsilon$ *for every* $i \in I$.

The elements of $\mathscr{R}$ will be called *rectangles,* whereas the elements of $\operatorname{div}(Q)$, for $Q \in \mathscr{R}$, will be called the *subdivisions* of Q. Later, we shall also need the following.

DEFINITION 3.2. *A rectangular system* $(\mathscr{R}, \operatorname{div})$ *is said to be full if for any* $Q \in \mathscr{R}$ *and any* $R_1, \ldots, R_m \in \mathscr{R}$ *with* $R_\nu \subseteq Q, 1 \leqslant \nu \leqslant m$, *there exists a subdivision* $(Q_i)_{i \in I}$ *of* Q *and, for each* ν, *a subset* I_ν *of* I *such that* $(Q_i)_{i \in I_\nu}$ *is a subdivision of* R_ν.

Let $(\mathscr{R}, \operatorname{div})$ be a rectangular system on $\mathscr{X}$. A complex valued function φ defined on $\mathscr{R}$ is said to be *additive* if $\varphi(Q) = \sum_{i \in I} \varphi(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q. Similarly, a real-valued function s defined on $\mathscr{R}$ is called *subadditive* if $s(Q) \leqslant \sum_{i \in I} s(Q_i)$ for any subdivision $(Q_i)_{i \in I}$ of Q. The function s is called *superadditive* if $-s$ is subadditive.

DEFINITION 3.3. *Let* φ *be additive and* s *superadditive on* $\mathscr{R}$. *A subset* $A \subset \mathscr{X}$ *is said to be* (φ, s)*-negligible if, for any* $Q \in \mathscr{R}$ *and any* $\varepsilon > 0$, *there exist a subdivision* $(Q_i)_{i \in I}$ *of* Q *and a subset* J *of* I *so that* $Q_i \cap A = \varnothing$ *for any* $i \in I \setminus J$, *and such that*

$$\left| \sum_{i \in J} \varphi(Q_i) \right| \leqslant \sum_{i \in J} s(Q_i) + \varepsilon .$$

We are now in a position to state in precise terms the localization principle alluded to in the introduction.

THEOREM 3.4. *Let* $(\mathscr{R}, \operatorname{div})$ *be a rectangular system on the complete metric space* $\mathscr{X}$. *Also, let* φ *be an additive function on* $\mathscr{R}, s$ *a superadditive function on* $\mathscr{R}$, *and let* $A \subset \mathscr{X}$ *be a countable union of* (φ, s)*-negligible subsets of* $\mathscr{X}$. *The following conditions are equivalent:*

(1) $|\varphi(Q)| \leqslant s(Q)$ *for all* $Q \in \mathscr{R}$;

(2) there exists a positive, superadditive function t *on* $\mathscr{R}$ *so that* $|\varphi(Q)| \leqslant s(Q)$ *whenever* $t(Q) = 0$ *and such that, for any nested sequence of rectangles* $(Q_\nu)_\nu$ *having* $t(Q_\nu) > 0$ *for all* ν, *and* $\cap_\nu Q_\nu = \{a\}$ *for some* $a \in \mathscr{X} \setminus A$, *we have*

$$\liminf_{\nu} \frac{|\varphi(Q_\nu)| - s(Q_\nu)}{t(Q_\nu)} \leqslant 0 ;$$

(3) there exists a positive, superadditive function t *on* $\mathscr{R}$ *and, for any* $a \in \mathscr{X} \setminus A$ *and any* $\varepsilon > 0$, *an open neighborhood* $\mathscr{U}$ *of* a *in* $\mathscr{X}$ *such that*

$$|\varphi(Q)| \leqslant s(Q) + \varepsilon t(Q),$$

for any rectangle Q *included in* $\mathscr{U}$ *and containing* a.

Furthermore, the above conditions still remain equivalent with $|\varphi(\cdot)|$ *replaced by* φ.

Note the analogy of this result with the maximum principle from potential theory (the additive and subadditive functions correspond to the harmonic and subharmonic functions, respectively, whereas χ appears as a kind of ideal boundary of $\mathscr{R}$).

Let us also point out that for $t =$ constant, Theorem 3.4 is essentially a principle for passing from *local* to *global*, while for $t \neq$ constant a principle for passing from *infinitesimal* to *global.*

Proof of Theorem 3.4. Obviously, (1) implies (2). Moreover, a straightforward reasoning by contradiction shows that any function t satisfying the hypothesis (2) will automatically do for (3).

We are therefore left with (3) $\Rightarrow$ (1). Once again, we shall reason by contradiction. To this effect, assume that there exists a rectangle Q such that $|\varphi(Q)| > s(Q)$. In particular, this implies that $t(Q) > 0$. Now fix $\varepsilon > 0$, small enough so that

$$|\varphi(Q)| - s(Q) > \varepsilon t(Q),$$

and set $\varepsilon_\nu := (2^{-1} + 3^{-1-\nu})\varepsilon$. Let $A = \cup_{\nu=0}^{\infty} A_\nu$, where A_ν is a (φ, s)-negligible subset of $\mathscr{X}$ for each $\nu \in \mathbf{N}$. In particular, since A_0 is (φ, s)-negligible, there exist a subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I for which $Q_i \cap A_0 = \varnothing$, when $i \in I \setminus J$, and such that

$$\left| \sum_{i \in J} \varphi(Q_i) \right| \leqslant \sum_{i \in J} s(Q_i) + |\varphi(Q)| - s(Q) - \varepsilon t(Q). \tag{3.1}$$

Next we claim that we cannot have $|\varphi(Q_i)| \leqslant s(Q_i) + \varepsilon_0 t(Q_i)$ for all $i \in I \setminus J$. To prove the claim, we remark that since t is positive and superadditive, this would lead to

$$\begin{aligned} \sum_{i \in I\setminus J} |\varphi(Q_i)| &\leqslant \sum_{i \in I\setminus J} s(Q_i) + \varepsilon_0 \sum_{i \in I\setminus J} t(Q_i) \\ &\leqslant \sum_{i \in I\setminus J} s(Q_i) + \varepsilon_0 t(Q). \end{aligned} \tag{3.2}$$

In turn, since φ is additive and s superadditive, a simple combination of (3.1) and (3.2) would imply that $0 \leqslant \varepsilon_0 t(Q) - \varepsilon t(Q)$, which is a contradiction. Consequently, one can find an index $i_0 \in I \setminus J$ for which

$$|\varphi(Q_{i_0})| > s(Q_{i_0}) + \varepsilon_0 t(Q_{i_0}) .$$

Because Q_{i_0} and A_0 are disjoint it follows that it is possible to find a rectangle $R_0 \subseteq Q$ which does not intersect A_0, has $\operatorname{diam}(R_0) \leqslant 1$, and such that

$$|\varphi(R_0)| > s(R_0) + \varepsilon_0 t(R_0) .$$

Continuing this inductively, one can construct a sequence of nested rectangles $\{R_\nu\}_\nu$ such that R_ν does not meet A_ν, $\operatorname{diam}(R_\nu) \leqslant 2^{-\nu}$, and

$$|\varphi(R_\nu)| > s(R_\nu) + \varepsilon_\nu t(R_\nu) \geqslant s(R_\nu) + \frac{\varepsilon}{2} t(R_\nu) ,$$

for any $\nu \in \mathbf{N}$. But then $\cap_\nu R_\nu = \{a\}$ for some $a \in \mathscr{X} \setminus A$ and this contradicts (3). The proof is finished. $\square$

We shall also use the following version of the Theorem 3.4.

THEOREM 3.5. *Let* $\mathscr{X}, (\mathscr{R}, \operatorname{div}), A, \varphi, s$ *be as in Theorem 3.4 and assume that* t *is a positive, superadditive function on* $\mathscr{R}$. *Then, the following conditions are equivalent:*

(1) for any relatively compact open subset Ω *of* $\mathscr{X}$ *there exists* $M > 0$ *such that* $|\varphi(Q)| \leqslant s(Q) + Mt(Q)$ *for all* $Q \in \mathscr{R}$ *with* $Q \subseteqq \Omega$;

(2) $|\varphi(Q)| \leqslant s(Q)$ *whenever* $t(Q) = 0$ *and for any nested sequence of rectangles* $\{Q_\nu\}_\nu$ *having* $t(Q_\nu) > 0$ *for all* ν, *and* $\cap_\nu Q_\nu = \{a\}$ *for some* $a \in \mathscr{X} \setminus A$, *we have*

$$\limsup_\nu \frac{|\varphi(Q_\nu)| - s(Q_\nu)}{t(Q_\nu)} < +\infty ;$$

(3) for any $a \in \mathscr{X} \setminus A$ *there exist an open neighborhood* $\mathscr{U}$ *of* a *in* $\mathscr{X}$ *and* $M > 0$ *such that*

$$|\varphi(Q)| \leqslant s(Q) + Mt(Q) ,$$

for any rectangle Q *included in* $\mathscr{U}$ *and containing* a.

The proof is quite similar to that of Theorem 3.4 and we omit it.

4. The global Stokes formula for simple Lipschitz domains in $\mathbf{R}^n$

A $(n-1)$-form u on $\mathbf{R}^n$ is said to be *uniformly locally $(n-1)$-integrable on* $\Omega \subseteq \mathbf{R}^n$ if it is locally $(n-1)$-integrable and, for any compact subset K of $\mathbf{R}^n$ and any $\varepsilon > 0$, there exists a positive $\delta = \delta(K, \varepsilon)$ such that

$$\left| \int_C u \right| < \varepsilon \tag{4.1}$$

whenever C is a $(n-1)$-dimensional Lipschitz submanifold C of $\mathbf{R}^n$ which is contained in $K \cap \Omega$ and has $\mu_{n-1}(C) < \delta$.

Examples include $(n-1)$-forms with locally bounded coefficients, or exhibiting isolated singularities of the type $\| x \|^{-\alpha}$, $\alpha < n-1$.

Let us recall the notion of simple Lipschitz domain introduced in the last part of Definition 1.1. The main result of this section is the following.

THEOREM 4.1. *Let Ω be a simple Lipschitz domain in $\mathbf{R}^n$, and let u be a compactly supported $(n-1)$-form in $\mathbf{R}^n$ which is uniformly $(n-1)$-locally integrable on $\mathbf{R}^n$. Assume that u is absolutely continuous on Ω and that the singular set*

$$\mathscr{S}(u) := (\bar{\Omega} \backslash \Omega) \cap \operatorname{supp} u$$

has $(n-1)$-dimensional Hausdorff measure zero.

Then, if u is integrable on $b\Omega$ and du (in the distribution sense) is integrable on Ω, we have

$$\int_{b\Omega} u = \iint_{\mathring{\Omega}} du \,.$$

To prove this theorem, we shall need an auxiliary lemma. Two Lipschitz domains Ω_1, Ω_2 in $\mathbf{R}^n$ will be called *almost transversal* if $\mu_{n-1}(b\Omega_1 \cap b\Omega_2) = 0$. Let Ω be a Lipschitz domain in $\mathbf{R}^n$ and let $\mathscr{R}$ stand for the collection of all rectangles of $\mathbf{R}^n$ which are almost transversal to Ω. Next, assume that u is a $(n-1)$-form compactly supported on $\mathbf{R}^n$, uniformly locally $(n-1)$-integrable on $\mathbf{R}^n$, and integrable on $b\Omega$. Also, let f be a locally integrable n-form on $\mathbf{R}^n$ and consider the complex-valued mapping φ defined on $\mathscr{R}$ by

$$\varphi(Q) := \int_{\mathring{Q} \cap b\Omega} U + \int_{\mathring{\Omega} \cap \partial Q} u - \iint_{Q \cap \Omega} f \,.$$

LEMMA 4.2. *Let* $\Omega, \mathscr{R}, u, f, \varphi$ *be as above and assume that* $\mathscr{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \operatorname{supp} u$ *has Hausdorff* $(n-1)$*-dimensional measure zero. Then the following hold.*

(1) $\mathscr{R}$ *together with the usual subdivisions is a full rectangular system on* $\mathbf{R}^n$.

(2) If P *is a* $\mathscr{R}$*-paved set and* $(Q_i)_{i \in I}$ *is a subdivision of* P, *then*

$$\sum_{i \in I} \varphi(Q_i) = \int_{\mathring{P} \cap b\Omega} u + \int_{\mathring{\Omega} \cap \partial P} u - \iint_{P \cap \Omega} f .$$

In particular, φ *is additive.*

(3) The set $\mathscr{S}(u)$ *is* $(\varphi, 0)$*-negligible.*

Proof. For each $k = 1, 2, \ldots, n$, let A_k be the collection of all $c \in \mathbf{R}$ having the property that

$$\mu_{n-1}(\{x = (x_1, \ldots, x_n) \in b\Omega; x_k = c\}) > 0 .$$

Since $\lambda_n(b\Omega) = 0$, it follows by Fubini's theorem that A_k has Lebesgue measure zero in $\mathbf{R}$ for any k.

Consider now $Q, R_1, \ldots, R_m \in \mathscr{R}$ such that $R_\nu \subseteq Q$ for all ν. Let $(a_1, \ldots, a_n)$ be the origin of Q, and $(b_1, \ldots, b_n)$ the end-point of Q. Similarly, for each ν, $(a_1^\nu, \ldots, a_n^\nu)$ will stand for the origin of R_ν, whereas $(b_1^\nu, \ldots, b_n^\nu)$ will denote the end-point of R_ν. The almost transversality hypothesis implies that $a_k, b_k, a_k^\nu, b_k^\nu \in \mathbf{R} \setminus A_k$ for all ν, k.

Now, since $\lambda_1(A_k) = 0$, for any a priory given $\varepsilon > 0$, we can select a finite sequence of real numbers $x_{k,\alpha_k}^\nu \in \mathbf{R} \setminus A_k$, $\alpha_k = 0, \ldots, p_k$, such that

$$a_k = x_{k,0}^\nu < \cdots < x_{k,p_k}^\nu = b_k ,$$
$$|x_{k,\alpha_{k-1}}^\nu - x_{k,\alpha_k}^\nu| \leqslant \varepsilon n^{-1/2} ,$$

and, finally, so that a_k^ν and b_k^ν are among the numbers $\{x_{k,\alpha_k}^\nu\}_{\alpha_k}$. It is then easy to see that, for ε sufficiently small, the rectangles

$$Q_{(\alpha_1, \ldots, \alpha_n)} := \prod_{k=1}^{n} [x_{k,\alpha_{k-1}}, x_{k,\alpha_k}], \quad \text{with } 1 \leqslant \alpha_k \leqslant p_k ,$$

form an elementary subdivision of Q which contains a subdivision of R_ν for each $1 \leqslant \nu \leqslant m$. This completes the proof of (1).

Going further, (2) is immediate in the case in which the family $(Q_i)_{i \in I}$ comes from an elementary subdivision of a larger rectangle containing P. Thus, the general case then easily follows from this and (1).

Next we turn our attention to (3). Fix $Q \in \mathscr{R}, K$ a compact subset of $\Omega \setminus \mathscr{S}(u)$ and $\varepsilon > 0$. Since $\mathscr{S}(u)$ has $(n-1)$-dimensional Hausdorff measure zero, it is thus possible to select finitely many rectangles $R_1, \ldots, R_m \in \mathscr{R}$ which do not intersect K, their interiors cover $Q \cap \mathscr{S}(u)$, and such that

$$\sum_{\nu=1}^{m} \mu_{n-1}(\partial R_\nu) < \varepsilon .$$

Then $P := \cup_\nu (Q \cap R_\nu)$ is a $\mathscr{R}$-paved set contained in Q which does not intersect K and has the property that $\mu_{n-1}(\partial P) < \varepsilon$. Since $\mathscr{R}$ is full, we can find an elementary subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I for which $P = \cup_{i \in J} Q_i$. In particular, we note that this implies $Q_i \cap \mathscr{S}(u) = \varnothing$ for $i \in I \setminus J$. Using (2), we can write

$$\sum_{i \in J} \varphi(Q_i) = \int_{\overset{\circ}{P} \cap b\Omega} u + \int_{\overset{\circ}{\Omega} \cap \partial P} u - \iint_P f .$$

Now, since u is integrable on $b\Omega$ and f is integrable on Ω, the first and the third terms from above can be made arbitrarily small by choosing K large enough. Furthermore, by taking ε sufficiently small and using the fact that u is uniformly locally $(n-1)$-integrable, the second term can also be made arbitrarily small. The proof of the lemma is therefore finished. □

Proof of Theorem 4.1. Since in the conclusion of the theorem u intervenes only through its values on Ω, there is no loss of generality assuming that $u = 0$ on $\mathbf{R}^n \setminus \bar{\Omega}$, i.e. that $\operatorname{supp} u \subseteq \bar{\Omega}$ (note that this does not alter the hypotheses either). We set $f := du$ in $\overset{\circ}{\Omega}$, zero in $\mathbf{R}^n \setminus \overset{\circ}{\Omega}$, and adopt the notation introduced in Lemma 4.2. Clearly, it is enough to prove that $\varphi(Q) = 0$ for any $Q \in \mathscr{R}$. First, let us observe that from (the proof of) Theorem 1.3 this is immediate for rectangles of the following two types:

(1) $Q \subset \overset{\circ}{\Omega}$ or $u = 0$ on Q;

(2) after suitably permuting the coordinates in $\mathbf{R}^n$,

$$Q \cap \Omega = \{x = (x', x_n); x' \in Q' \quad \text{and} \quad a_n \leqslant x_n \leqslant \theta(x') < b_n\} ,$$

where $Q = Q' \times [a_n, b_n]$ and $\theta : \mathbf{R}^{n-1} \to (a_n, b_n)$ is a Lipschitz function. On the other hand, the compact set $\mathscr{S}(u)$ has zero μ_{n-1}-measure and, hence, by Lemma 4.2, is $(\varphi, 0)$-negligible. Consequently, using Theorem 3.4 with $s = t = 0$, it suffices to show that any point $a \in b\Omega$ has an open neighborhood $\mathscr{U}$ in $\mathbf{R}^n$ such that $\varphi(R) = 0$ for all rectangles $R \in \mathscr{R}$ included in $\mathscr{U}$ and containing a. By possibly relabeling the coordinates first, we can find an open rectangle U in $\mathbf{R}^n$ and a Lipschitz function $\theta: \mathbf{R}^{n-1} \to \mathbf{R}$ such that

$$U \cap \Omega = U \cap \{x = (x', x_n); x_n \leqslant \theta(x')\} .$$

Now let $R = R' \times [a_n, b_n] \in \mathscr{R}$ be a fixed rectangle contained in U, where R' is a rectangle in $\mathbf{R}^{n-1}$ and $a_n, b_n \in \mathbf{R}$, $a_n < b_n$. Denote by $\mathscr{R}'$ the collection of all rectangles Q' from $\mathbf{R}^{n-1}$ which are contained in R', having $p(Q') \leqslant p(R') + 1$ and such that $Q' \times [a_n, b_n] \in \mathscr{R}$. Then, with the usual subdivisions, $(\mathscr{R}', \text{div})$ becomes a rectangular system on R'.

Next, we introduce the mapping $\psi: \mathscr{R}' \to \mathbf{C}$ by setting

$$\psi(Q') := \varphi(Q' \times [a_n, b_n]), \quad Q' \in \mathscr{R}'$$

Thus, everything comes down to proving that ψ vanishes identically on $\mathscr{R}'$. Let us consider the following compact set in $\mathbf{R}^n$:

$$A' := R' \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)) .$$

If a rectangle $Q' \in \mathscr{R}'$ does not meet A', then the rectangle $Q' \times [a_n, b_n] \in \mathscr{R}$ is either of type (1) or (2) from above, so that, at any rate, $\psi(Q') = 0$.

Since φ is additive, so is ψ and, by the equivalence (1) $\Leftrightarrow$ (3) in Theorem 3.4 with $s = t = 0$, it suffices to prove that A' is $(\psi, 0)$-negligible. To this end, let $Q' \in \mathscr{R}'$ and let $(Q_i')_{i \in I}$ be a subdivision of Q' such that $\delta_i := \operatorname{diam}(Q_i') \leqslant \delta$, for all i, for some positive δ. We also introduce

$$J := \{i \in I; Q_i' \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)) \neq \varnothing\} .$$

For each $i \in J$ we have that at least one of the sets $Q_i' \cap \theta^{-1}(a_n)$, $Q_i' \cap \theta^{-1}(b_n)$ is empty provided δ is sufficiently small. Assuming that this is the case, we set

$$Q_i := Q_i' \times [a_n, a_n + \delta_i M]$$

if $Q_i' \cap \theta^{-1}(a_n) \neq \varnothing$, and

$$Q_i := Q_i' \times [b_n - \delta_i M, b_n] ,$$

if $Q_i' \cap \theta^{-1}(b_n) \neq \varnothing$. Here M stands for the (essential) supremum of $|\nabla \theta|$ on R'. Then $P := \cup_{i \in J} Q_i$ is a $\mathscr{R}$-paved set having

$$\mu_{n-1}(\partial P) \leqslant C \sum_{i \in J} \mu_{n-1}(Q_i') \tag{4.2}$$

for some positive constant C depending exclusively on θ and R'. Furthermore, as $\varphi(Q) = 0$ for any Q of the types (1)-(2) described above, and since φ is additive, it follows that $\psi(Q_i') = \varphi(Q_i)$ for any $i \in J$. In particular,

$$\left| \sum_{i \in J} \psi(Q_i') \right| = |\varphi(P)| \leqslant \left| \int_{\overset{\circ}{\Omega} \cap \partial P} u \right| + \left| \int_{\overset{\circ}{P} \cap b\Omega} u \right| + \left| \iint_{P \cap \Omega} f \right| .$$

By (4.2), the uniformly local $(n-1)$-integrability of u, the integrability of u on $b\Omega$ and the integrability of f on Ω, the right hand side of the above equality can be made arbitrarily small, provided $\sum_{i \in J} \mu_{n-1}(Q_i')$ is sufficiently small. However, since A' has Lebesgue measure zero in $\mathbf{R}^{n-1}$, this can be readily taken care of and this completes the proof of the theorem. □

REMARK 4.3. As an inspection of the proofs shows, Theorem 4.1 and Lemma 4.2 continue to hold in the case when the locally $(n-1)$-integrable form u is *uniformly* $(n-1)$-integrable only in a small neighborhood of $\mathcal{S}(u)$.

5. THE GLOBAL FORM OF THE STOKES FORMULA ON C^1 MANIFOLDS

In this section we shall present a coordinate free version of the main result of section 4. Throughout this section, we let M be a fixed, oriented, Hausdorff, differentiable manifold of class C^1, and real dimension n.

DEFINITION 5.1. *A subset* Ω *of* M *is called a* C^1 *domain if for any* $a \in \Omega \backslash \overset{\circ}{\Omega}$, *there exist an open neighborhood* U *of* a *in* M *and a* C^1 *diffeomorphism* $f = (f_1, f_2, \ldots, f_n)$ *of* U *onto an open neighborhood* V *of the origin in* $\mathbf{R}^n$, *such that*

$$U \cap \Omega = \{x \in U; f_n(x) \leqslant 0\} .$$

Clearly, the *border of the domain* $\Omega, b\Omega := \Omega \backslash \overset{\circ}{\Omega}$ is either the empty set or a $(n-1)$-dimensional C^1-submanifold of M assumed with the standard induced orientation. Note that a simple application of the implicit function theorem shows that any C^1 domain is also a Lipschitz domain in $\mathbf{R}^n$.

It is not difficult to see that the class of Lipschitz domains described in Definition 1.1 is not invariant under the action of bi-Lipschitz diffeomorphisms of $\mathbf{R}^n$. In particular, Theorem 4.1 cannot be reformulated invariantly. To remedy this, for the rest of this section we shall slightly adjust our previous definitions to the C^1 framework by carrying out the following simple modification. That is, whenever applicable, we shall replace "Lipschitz embedding" by "*C^1-embedding*", i.e. Lipschitz embeddings which are C^1 functions. Note that, in particular, the condition (1.1) is in this case equivalent with the continuity of the functions

$$\frac{\partial h(s,x)}{\partial x_i} : S \times \omega \to \Omega, \quad i = 1, 2, \ldots, n-1 .$$

Assuming this modification, all the previously introduced notions become invariant to C^1 diffeomorphisms and, hence, meaningful on C^1 manifolds. More specifically, we make the following.

DEFINITION 5.2. *Let Ω be a C^1 domain of M. A $(n-1)$-form u is said to be absolutely continuous (uniformly $(n-1)$-locally integrable) on Ω if for any point $P \in \Omega$ there exists a local coordinate map $h: U \to \mathbf{R}^n$ of M with $P \in U$ such that $(h^{-1})^* u$ is absolutely continuous (uniformly $(n-1)$-locally integrable, respectively) on $h(U \cap \Omega)$.*

Let u and f be locally integrable forms on M, having degrees $(n-1)$ and n, respectively. Recall that $du = f$ on a open set Ω of M *in the distribution sense,* if for any $\varphi \in C_0^1(\Omega)$,

$$\int_M d\varphi \wedge u = -\int_M \varphi f .$$

THEOREM 5.3. *Let Ω be a C^1 domain of M, and u a $(n-1)$-form compactly supported in M. Assume that u is uniformly $(n-1)$-locally integrable and absolutely continuous on Ω, and that the singular set*

$$\mathscr{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \sup u$$

has $(n-1)$-dimensional Hausdorff measure zero.

If u is integrable on $b\Omega$ and du (taken in the sense of distribution theory) is integrable on Ω, then

$$\int_{b\Omega} u = \iint_{\mathring{\Omega}} du .$$

Proof. Using a smooth partition of unity and then working in local coordinates we can assume that $M = \mathbf{R}^n$. In this case, the conclusion is provided by Theorem 4.1. $\square$

Note that, here again it suffices to have the "uniform" part of the local $(n-1)$-integrability condition for u fulfilled only on a small neighborhood of $\mathscr{S}(u)$ (cf. also Remark 4.3).

DEFINITION 5.4. *A closed subset A of M is said to have an almost regular boundary if A coincides with the closure of its interior and if there exist a family $(S_i)_{i \in I}$ of C^1 submanifolds of M and a locally finite family $(C_i)_{i \in I}$ of compact subsets of M such that:*

(1) $C_i \subset S_i$, for any $i \in I$, and $\mathring{C}_i \cap \mathring{C}_j = \varnothing$, for all $i \neq j$ (the interiors are taken in S_i and in S_j, respectively);

(2) $C_i \cap C_j$ has $(n-1)$-dimensional Hausdorff measure zero for all $i \neq j$;

(3) $\partial A = \cup_{i \in I} C_i$.

Note that if A has an almost regular boundary, then

$$\Omega := \mathring{A} \cup \left(\bigcup_{i \in I} \mathring{C}_i \right)$$

(the interior of A is taken in M) is a C^1 domain with border $b\Omega = \cup_{i \in I} \mathring{C}_i$. If u is an integrally continuous $(n-1)$-form on M, it follows that u is integrable on each oriented submanifold $\mathring{C}_i$ (with the standard orientation induced by $\mathring{A}$). Since ∂C_i has zero measure in S_i, we can define

$$\int_{\partial A} u := \sum_{i \in I} \int_{\mathring{C}_i} u$$

whenever $A \cap \operatorname{supp} u$ is compact. Hence, without further proof, we can state the following.

THEOREM 5.5. *Let A be a subset of M with an almost regular boundary. If u is a $(n-1)$-form which is uniformly $(n-1)$-locally integrable on M, absolutely continuous on M, and for which $A \cap \operatorname{supp} u$ is compact, then u is integrable on ∂A, du is integrable on $\mathring{A}$ and*

$$\int_{\partial A} u = \iint_{\mathring{A}} du .$$

6. Tests for the equalities $du = f$ and $\bar{\partial} u = f$ in the weak sense

Let $\Omega \subseteq \mathbf{R}^n$ be a fixed open set in $\mathbf{R}^n$. In this section we let $\mathscr{R}(\Omega)$ denote the collection of all rectangles Q contained in Ω and having $p(Q) \leqslant c_0$, for a fixed real number $1 < c_0 < +\infty$.

First, we shall give a coordinate free definition of the exterior differentiation operation. This builds on the classical work of Pompeiu [Po2] for the case $n = 2$ (cf. also the results in §7).

Definition 6.1. *A $(n-1)$-form u which is locally $(n-1)$-integrable on Ω is said to be exteriorly differentiable at $a \in \Omega$ if the limit*

$$c := \lim_{Q \downarrow a} \frac{1}{\lambda_n(Q)} \int_{\partial Q} u$$

exists in $\mathbf{C}$. More specifically, we assume that there exists a complex number c so that, for any $\varepsilon > 0$, there exists an open neighborhood $U \subseteq \Omega$ of a such that

$$\left| \int_{\partial Q} u - c\lambda_n(Q) \right| < \varepsilon \lambda_n(Q) \,,$$

for all $Q \in \mathscr{R}(\Omega)$, $Q \subset U$.

We then set $u'(a) := c$ and $du_a := c\pi_1 \wedge \cdots \wedge \pi_n$, where $\pi_1, \ldots, \pi_n$ are the canonical coordinate projections of $\mathbf{R}^n$.

For $n = 1$, u' becomes the usual derivative of the function u. Our next theorem collects several exterior differentiability criteria for $(n-1)$-forms.

Theorem 6.2. *Let $u = \sum_{i=1}^{n} (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ be a locally $(n-1)$-integrable form on Ω.*

(1) If the function $u_i, i = 1, \ldots, n$ are differentiable at $a \in \Omega$, then u is exteriorly differentiable at a and

$$u'(a) = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}(a) \,.$$

(2) If u satisfies one of the equivalent conditions in Theorem 1.3, then u is exteriorly differentiable at almost every point of Ω and $u' \in L^1(\Omega, \text{loc})$. In particular, this is the case if u is absolutely continuous or integrally Lipschitz on Ω.

Proof. By hypotheses, there exist some numbers $c_{ij} \in \mathbf{R}$ and some functions ξ_i which are continuous and vanish at a, such that

$$u_i(x) = u_i(a) + \sum_{j=1}^{n} c_{ij}(x_j - a_j) + \xi_i(x)\,\|x - a\|, \quad i = 1, 2, \ldots, n\,.$$

A straightforward computation then yields $u'(a) = \sum_{i=1}^{n} c_{ii}$.

The second part of the conclusion follows directly from Theorem 1.3 and Lebesgue's differentiation theorem. □

Now we consider a p-form $u = \sum'_{|I|=p} u_I dx^I$ on $\Omega, 0 \leqslant p \leqslant n-1$. Here $\sum'$ indicates that the sum is performed over the set of all strictly increasing multi-indices I of length p, i.e. all ordered p-tuples of the form $I = (i_1, \ldots, i_p)$, with $1 \leqslant i_1 < \cdots < i_p \leqslant n$. Also, dx^I stands for $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ if $I = (i_1, \ldots, i_p)$. For each strictly increasing multi-index J of length $p+1$ we introduce the $(n-1)$-form

$$u^J := \sum_{i=1}^{n} (-1)^{i-1} \left(\sum_{|I|=p}{}' \varepsilon_J^{iI} u_I \right) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\,.$$

Here $\varepsilon_J^{iI} = 0$ unless $\{i\} \cup I = J$, in which case ε_J^{iI} is the sign of the permutation taking iI, the concatenation of $\{i\}$ and I, onto J.

The forms u^J will be called the *$(n-1)$-forms associated to u*. Since, clearly, the application

$$u \mapsto \{u^J;\, |J| = p+1\}$$

is one-to-one, we can represent a given differential form either by its coefficients, or by the $(n-1)$-forms associated to it. In fact, for $p = n-1$, the functions $u^J, |J| = n$, are precisely the coefficients of the form u. Furthermore, one can easily check that u is locally integrable if and only each of its associated $(n-1)$-forms is locally integrable.

It is natural to use the associated $(n-1)$-forms to extend the concepts already defined for $p = n-1$ to the general case of p-forms, $p \leqslant n-1$. More specifically, a p-form is called *locally $(n-1)$-integrable, exteriorly differentiable at a,* etc, if all its associated $(n-1)$-forms have that particular property. In the case when u^J's are exteriorly differentiable at $a \in \Omega$, we also set

$$du_a := \sum_{|J|=p+1}{}' (u^J)'(a)\,\pi^J\,,$$

where $\pi^J := \pi_{j_1} \wedge \cdots \wedge \pi_{j_{p+1}}$ if $J = (j_1, \ldots, j_{p+1})$.

Suppose now that $u = \sum_{i=1}^{n}(-1)^{i-1} u_i\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ is a $(n-1)$-form on Ω. For $0 \leqslant r \leqslant 1, x \in \Omega$, and $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$, we set

$$\omega_x(u, \varepsilon) := \sup_{y \neq z \in B_\varepsilon(x)} \frac{\| u(y) - u(z) \|}{\| y - z \|^r} .$$

where $B_\varepsilon(x) \subset \mathbf{R}^n$ is the ball of radius ε centered at x and $u(x)$ is identified with the point $(u_1(x), \ldots, u_n(x))$ of $\mathbf{R}^n$, etc.

For a $(n-1)$-form u on Ω and a subset $C \subset \Omega$, we consider the following conditions:

Condition (α). $\mu_{n-1}(C) = 0$, u is locally $(n-1)$-integrable on Ω and uniformly locally $(n-1)$-integrable on a neighborhood of C.

Condition (β). There exists some $0 < r \leqslant 1$ such that $\mu_{n+r-1}(C) = 0$, u is uniformly locally $(n-1)$-integrable on Ω and has the property that

$$\omega_x(u, \varepsilon) = O(1), \quad \text{as } \varepsilon \to 0 , \tag{6.1}$$

at each point x of Ω outside some closed, μ_{n-1}-negligible set $A \subset \Omega$.

Condition (γ). There exists some $0 \leqslant r < 1$ such that $\mu_{n+r-1}(C) < +\infty$, u is uniformly locally $(n-1)$-integrable on Ω and has the property that

$$\omega_x(u, \varepsilon) = o(1), \quad \text{as } \varepsilon \to 0 , \tag{6.2}$$

at each point x of Ω outside some closed, μ_{n-1}-negligible set $A \subset \Omega$.

The main results of this section are the following.

THEOREM 6.3. *Consider a complex-valued, locally $(n-1)$-integrable p-form u on Ω. Let $(C_\nu)_\nu$ be an at most countable collection of closed subsets of Ω such that, for each ν and each associated $(n-1)$-form u^J of u, the pair (u^J, C_ν) satisfies one of the conditions (α)-(γ) stated above. Furthermore, assume that for any multiindex J*

$$\limsup_{Q \downarrow x} \frac{1}{\lambda_n(Q)} \left| \int_{\partial Q} u^J \right| < +\infty \tag{6.3}$$

at any $x \in \Omega \setminus (\cup_\nu C_\nu)$.

Then, for each J, the restriction of u^J to any relatively compact open subdomain of Ω is integrally Lipschitz. In particular, u is exteriorly differentiable almost everywhere on Ω.

THEOREM 6.4. *Let u be a complex-valued locally integrable p-form which is locally $(n-1)$-integrable on Ω and let $(C_\nu)_\nu$ be a as in*

Theorem 6.3. Also, set $A := \cup_\nu C_\nu$ *and consider a complex-valued* $(p+1)$*-form* f *in* $L^1(\Omega, \mathrm{loc})$. *Furthermore, assume that at least one of the following conditions is fulfilled:*

(1) A *is closed,* u *is integrally continuous on* $\Omega \setminus A$ *and* $du = f$ *in the distribution sense on* $\Omega \setminus A$*;*

(2) u *is exteriorly differentiable on* $\Omega \setminus A$ *and* $du_x = f_x$ *at each point* $x \in \Omega \setminus A$.

Then $du = f$ *in the distribution sense on* Ω.

REMARK 6.5. For integrally continuous forms u such that the limit in (6.3) vanishes for each J, Theorem 6.3 gives sufficient conditions for the equality $du = 0$ to hold in the distribution sense on Ω.

Moreover, in the case $f = 0$, Theorem 6.4 furnishes tests for a p-form to be closed, too. Theorem 6.3 also gives absolute continuity criteria for integrally continuous forms. In turn, these can be used to further improve the main results of §4 and §5.

The proofs of these theorems will be accomplished in a series of lemmas.

LEMMA 6.6. *Let* u *be a locally* $(n-1)$*-integrable* $(n-1)$*-form on* Ω, f *a locally integrable* n*-form on* Ω, *and let*

$$\varphi(Q) := \int_{\partial Q} u - \iint_Q f, \tag{6.4}$$

for $Q \in \mathscr{R}(\Omega)$. *Also, let* C *be a closed subset of* Ω. *If the pair* (u, C) *fulfills one of the conditions* (α)-(γ) *stated above, then the set* C *is* $(\varphi, 0)$*-negligible.*

Proof. If (α) is the fulfilled condition, then the statement follows from an obvious variant of Lemma 4.2, (3). To complete the proof in the remaining cases, let us consider $0 \leqslant r \leqslant 1$ such that C has finite $(n+r-1)$-dimensional Hausdorff measure. Also, let $A \subset \Omega$ with $\mu_{n-1}(A) = 0$ be the exceptional set appearing in the statement of the conditions (β) and (γ). Finally, we fix a rectangle $Q \in \mathscr{R}(\Omega)$ and two small numbers $\varepsilon, \delta > 0$.

Consider now two paved sets $P, R \subseteq Q$ such that $\mathring{Q} \cap C \subset \mathring{P}$ and $\mathring{Q} \cap A \subset \mathring{R}$. Without any loss of generality we can assume that $0 < \varepsilon < \operatorname{dist}(A, \partial P)$ and that $\mu_{n-1}(\partial R) < \delta$. We can also assume that there exist finitely many cubes $R_1, \dots, R_m$ with diameters inferior to ε so that $(R_\nu)_{\nu=1}^m$ is a subdivision of $P \setminus \overset{\circ}{R}$ and such that

$$\sum_{\nu=1}^{m} \operatorname{diam}(R_\nu)^{n+r-1} \leqslant \mu_{n+r-1}(C) + \varepsilon .$$

Next, let $(Q_i)_{i \in I}$ be a subdivision of Q such that $(Q_i)_{i \in I_1} = (R_\nu)_{\nu=1}^m$ for some $I_1 \subseteq I$ and that, for some $I_2 \subseteq I$, $(Q_i)_{i \in I_2}$ is a subdivision of R. We set $J := I_1 \cup I_2$. As a consequence, $Q_i \cap A = \varnothing$ for each $i \notin J$. Also,

$$\sum_{i \in J} \varphi\ (Q_i) = \sum_{\nu=1}^{m} \int_{\partial R_\nu} u + \int_{\partial R} u - \iint_{P \cup R} f .$$

Going further, for $\nu = 1, \dots, m$ we fix some points $x_\nu \in R_\nu$ and set

$$u(x_\nu) := \sum_{i=1}^{n} (-1)^{i-1} u_i(x_\nu)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n .$$

Then

$$\left| \int_{\partial R_\nu} u \right| = \left| \int_{\partial R_\nu} (u - u(x_\nu)) \right| \leqslant \sum_{\sigma} \left| \int_{\sigma} (u - u(x_\nu)) \right| ,$$

where the sum runs over the faces of R_ν. For instance, if σ is a face of R_ν on which $x_1 =$ constant, then

$$\left| \int_{\sigma} (u_1 - u_1(x_\nu))\, dx_2 \wedge \cdots \wedge dx_n \right| \leqslant \sup_{x \in \sigma} | u_1(x) - u_1(x_\nu) |\, \mu_{n-1}(\sigma) .$$

All in all, we get that

$$\left| \int_{\partial R_\nu} u \right| \leqslant c_n \omega_{x_\nu}(u, \varepsilon) \operatorname{diam}(R_\nu)^{n+r-1} ,$$

for some positive constant c_n depending solely on n. Adding up in ν we obtain

$$(6.5) \qquad \sum_{\nu=1}^{m} \left| \int_{\partial R_\nu} u \right| \leqslant c_n(\mu_{n+r-1}(C) + \varepsilon) \max_{1 \leqslant \nu \leqslant m} \omega_{x_\nu}(u, \varepsilon) .$$

Now, given $\theta > 0$, there exist $\varepsilon_0, \delta_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$ we have $| \iint_{P \cup R} f | < \theta / 3$. Also, if δ_0 is sufficiently small, from the uniform locally $(n-1)$-integrability of u we infer that $| \int_{\partial R} u | < \theta / 3$.

At this point, we fix ε_0, δ_0 and, by (6.1) (or 6.2), respectively), conclude that $\omega_{x_\nu}(u, \varepsilon) = O(1)$ (or $o(1)$, respectively) as $\varepsilon \to 0$, uniformly in ν. Using this, (6.4) and the assumptions concerning the size of $\mu_{n+r-1}(C)$, we get $|\int_{\partial(P \setminus \mathring{R})} u| < \theta/3$, provided ε is small enough.

Summarizing, for ε and δ as above, we see that $|\sum_{i \in J} \varphi(Q_i)| < \theta$, and the conclusion follows. □

LEMMA 6.7. *Let u be a $(n-1)$-form which is locally $(n-1)$-integrable on Ω and has real-valued coefficients, and let $(C_\nu)_\nu$ be an at most countable collection of closed subsets of Ω such that each pair (u, C_ν) satisfies one of the conditions $(\alpha)-(\gamma)$. Set $A := \cup_\nu C_\nu$ and let f be a locally integrable n-form on Ω, also having real-valued coefficients.*

If u is exteriorly differentiable on $\Omega \setminus A$ and $u'(x) \leqslant f(x)$ for all $x \in \Omega \setminus A$, then

$$\int_{\partial Q} u \leqslant \iint_Q f \tag{6.6}$$

for any $Q \in \mathscr{R}(\Omega)$.

Proof. Let us first assume that f is lower semi-continuous on Ω. We shall verify the condition (2) in Theorem 3.4 for the additive functions φ introduced in (6.3), and $t := \lambda_n$. To this effect, let us fix $a \in \Omega \setminus A$ and consider a nested sequence of rectangles $(Q_\nu)_\nu$ such that $\cap_\nu Q_\nu = \{a\}$. Since u is exteriorly differentiable at a and since f is lower semi-continuous it follows that

$$\liminf_\nu \frac{1}{\lambda_n(Q_\nu)} \int_{\partial Q_\nu} U = u'(a) \leqslant f(a) \leqslant \limsup_\nu \frac{1}{\lambda_\nu(Q_\nu)} \iint_{Q_\nu} f.$$

Consequently,

$$\liminf_\nu \frac{\varphi(Q_\nu)}{\lambda_n(Q_\nu)} \leqslant 0$$

and the conclusion is provided in this case by the equivalence (1) ⇔ (2) in Theorem 3.4.

Finally, as

$$\iint_Q f = \inf\left\{\iint_Q g; g \text{ lower semi-continuous and } \geqslant f\right\},$$

the general case obviously reduces to the one just considered. □

LEMMA 6.8. *Suppose that* $u, f, A,$ *are as in the first part of Lemma 6.6. In addition, assume that at least one of the following two conditions holds:*

(1) A *is closed and* $\int_{\partial Q} u = \iint_Q f$ *for any* $Q \in \mathscr{R}(\Omega)$ *such that* $Q \cap A = \varnothing$;

(2) u *is exteriorly differentiable on* $\Omega \backslash A$ *and* $du_x = f_x$ *for all* $x \in \Omega \backslash A$.

Then

$$\int_{\partial Q} u = \iint_Q f$$

for any $Q \in \mathscr{R}(\Omega)$.

Proof. In the first case the assertion follows directly from Lemma 6.6 and Theorem 3.4. As for the second one, the conclusion is immediately seen from Lemma 6.6. □

LEMMA 6.9. *Consider* $f = \sum'_{|J|=p+1} f_J dx^J$ *a locally integrable* $(p+1)$*-form on* Ω, *and let* u *be a locally integrable* p*-form on* Ω. *Then* $du = f$ *in the distribution sense if and only if* $du^J = f_J dx_1 \wedge \cdots \wedge dx_n$ *in the distribution sense for any* $J, |J| = p+1$.

Proof. For any smooth form v and for any $|J| = p+1$, a routine calculation shows that

$$dv^J = (dv)_J dx_1 \wedge \cdots \wedge dx_n .$$

The general case then follows from this observation and a standard regularization technique. □

Now we are ready to present the proofs of the main results of this section.

Proof of Theorem 6.3. The conclusions of the theorem are readily seen from Lemma 6.6, Theorem 3.5 and Theorem 6.2. □

Proof of Theorem 6.4. Using Lemma 6.8 one can reduce matters to $p = n-1$, in which case the theorem follows from Lemma 6.7 and Theorem 1.3. □

In the last part of this section we shall present similar results for the usual $\bar{\partial}$ operator acting on differential forms. Let $\Omega \subset \mathbf{C}^n$ be an open set, and let

$$u = \sum_{|I|=p}{}' \sum_{|K|=q}{}' u_{I,K}\, dz^I \wedge d\bar{z}^K$$

be a (p, q)-form on Ω, $0 \leqslant p \leqslant n$, $0 \leqslant q \leqslant n-1$. For any multi-indices I, J with $|I| = p$ and $|J| = q+1$, we set

$$u^{I,J} := (-1)^{n+p} \sum_{j=1}^{n} (-1)^{j-1} \left(\sum_{I,J}{}' \varepsilon_J^{jK} u_{I,K} \right) dz^{\{1,2,\ldots,n\}} \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots d\bar{z}_n.$$

The forms $u^{I,J}$ are called the $(n, n-1)$-*forms associated to* u. The concepts of integral continuity, etc, are introduced for (p, q)-forms as in the real case. We have the following.

THEOREM 6.10. *Let u be a locally integrable, complex-valued form of type (p, q), which is also locally $(n-1)$-integrable on an open subset Ω of $\mathbf{C}^n$. Let $(C_\nu)_\nu$ be a sequence of closed subsets of Ω such that each pair $(u^{I,J}, C_\nu)$ satisfies one of the conditions (α)-(γ). Also, let $A = \cup_\nu C_\nu$ and let f be a locally integrable form of type $(p, q+1)$ on Ω.*

Assume that at least one of the following conditions is valid:

(1) A is closed, u is integrally continuous on $\Omega \backslash A$ and $\bar{\partial} u = f$ in the distribution sense on $\Omega \backslash A$;

(2) u is exteriorly differentiable on $\Omega \backslash A$ and $\bar{\partial} u_x = f_x$ at each point $x \in \Omega \backslash A$.

Then $\bar{\partial} u = f$ in the distribution sense on Ω.

The proof is completely similar to the proof of the Theorem 6.4, hence omitted.

REMARK 6.11. For $f = 0$ we obtain tests for a (p, q)-form to be $\bar{\partial}$-closed, and for $p = q = 0$ tests for a function u to be holomorphic. The latter are well-known and due to Pompeiu [Po1] in the case $n = 1$. Our theorem also extends the holomorphy tests of [BM] and [Shi] in the case $n \geqslant 2$. Note that for $n = 2$, $p = q = 0$, $A = \varnothing$ and $f = 0$, we obtain the classical Goursat lemma.

Before we conclude this section, let us note that Theorem 6.3 naturally extends to the several complex variable setting and that this can also be used to obtain holomorphy criteria (cf. also [L]).

7. Some applications to hypercomplex function theory

The *Clifford algebra* associated with $\mathbf{R}^n$ endowed with the Euclidean metric is the enlargement of $\mathbf{R}^n$ to a unitary algebra $\mathscr{A}_n$ not generated (as an algebra) by any proper subspace of $\mathbf{R}^n$ and such that $x^2 = -|x|^2$, for any $x \in \mathbf{R}^n$. By polarization, this identity becomes

$$xy + yx = -2\langle x, y\rangle ,$$

for any $x, y \in \mathbf{R}^n$. In particular, if $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbf{R}^n$, one should have

$$e_j e_k + e_k e_j = -2\delta_{jk} .$$

Consequently, $e_j^2 = -1$ and $e_j e_k = -e_k e_j$ for any $j \neq k$. In particular, any element $u \in \mathscr{A}_n$ can be uniquely represented in the form $u = \sum_{k=0}^n \sum'_{|I|=k} u_I e_I$, with $u_I \in \mathbf{R}$, where e_I stands for the product $e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_k}$ if $I = (i_1, i_2, \ldots, i_k)$ (we make the convention that $e_\varnothing := 1$). More detailed accounts on these matters can be found in [BDS], [Mi].

The higher dimensional analogue of the form dz extensively used in the complex analysis of one variable is the $\mathscr{A}_n$-valued $(n-1)$-form

$$\omega := \sum_{j=1}^n (-1)^{j-1} e_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n .$$

For a compact Lipschitz domain Ω in $\mathbf{R}^n$, we let $d\sigma$ stand for the usual surface measure induced on $\partial\Omega$ by the Euclidean metric on $\mathbf{R}^n$, and let N denote the outward unit normal to Ω defined $d\sigma$-almost everywhere on $\partial\Omega$. As $\mathbf{R}^n \subset \mathscr{A}_n$, the vector valued function N can also be regarded as a $\mathscr{A}_n$-valued function on $\partial\Omega$. In fact, if ι denotes the inclusion of $\partial\Omega$ into $\mathbf{R}^n$, then

$$\iota^*(\omega) = N d\sigma .$$

An $\mathscr{A}_n$-valued function u defined on an open subset Ω of $\mathbf{R}^n$ is called *integrally continuous,* etc, provided the $\mathscr{A}_n$-valued $(n-1)$-form $u\omega$ has the corresponding property. Recall the generalized Cauchy-Riemann operator

$$D := \sum_{j=1}^n e_j \partial_j .$$

Let $\mathscr{R}(\Omega)$ be as defined at the beginning of §6. We also make the following definition.

DEFINITION 7.1.

(1) If $u = \sum_I' u_I e_I$ *is an* $\mathscr{A}_n$*-valued function defined on* $\Omega \subseteq \mathbf{R}^n$ *whose components* $(u_I)_I$ *are differentiable functions at a point* $a \in \Omega$, *then we define the action of* D *on* u *at* $a \in \Omega$ *by*

$$(Du)(a) := \sum_{i=1}^{n} \sum_I' \frac{\partial u_I}{\partial x_i}(a) e_i e_I .$$

(2) If u *and* f *are two locally integrable* $\mathscr{A}_n$*-valued functions on* Ω, *then we say that* $Du = f$ *in the distribution sense on* Ω *provided*

$$\iint_\Omega (D\psi) u dx = - \iint_\Omega \psi f dx$$

for any real-valued, smooth functions ψ, *compactly supported in* Ω.

(3) A locally $(n-1)$*-integrable,* $\mathscr{A}_n$*-valued function* u *is called Clifford differentiable at* $a \in \Omega$ *if the limit*

$$u'(a) := \lim_{Q \downarrow a} \frac{1}{\lambda_n(Q)} \int_{\partial Q} N u \, d\sigma$$

exists in $\mathscr{A}_n$.

The solutions of the (generalized) Cauchy-Riemann equations $Du = 0$ are called *monogenic functions*.

The theorems we are about to describe now are more or less immediate corollaries of the results obtained so far and we shall omit the proofs.

THEOREM 7.2. *Let* u *be a integrally continuous* $\mathscr{A}_n$*-valued function on the open set* Ω *of* $\mathbf{R}^n \subset \mathscr{A}_n$. *The following are equivalent.*

(1) There exists $f \in L^1_{\text{loc}}(\Omega, \mathscr{A}_n)$ *such that*

$$\int_{\partial Q} N u \, d\sigma = \iint_Q f dx$$

for any $Q \in \mathscr{R}(\Omega)$.

(2) There exists a real-valued, positive function $g \in L^1(\Omega, \text{loc})$ *such that*

$$\left| \int_{\partial Q} N u \, d\sigma \right| \leqslant \iint_Q g \, dx$$

for any $Q \in \mathscr{R}(\Omega)$.

(3) For any $Q \in \mathscr{R}(\Omega)$ *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\sum_{i \in J} \left| \int_{\partial Q_i} N_i u \, d\sigma_i \right| \leqslant \varepsilon ,$$

for any subdivision $(Q_i)_{i \in I}$ *of* Q *and any* $J \subseteq I$ *such that* $\sum_{i \in J} \lambda_n(Q_i) \leqslant \delta$.

(4) The function u *is Clifford differentiable almost everywhere on* Ω, u' *is locally integrable on* Ω *and*

$$\int_{\partial Q} N u \, d\sigma = \iint_{Q} u' \, dx$$

for any $Q \in \mathscr{R}(\Omega)$.

(5) The function u *is Clifford differentiable almost everywhere on* Ω, u' *is locally integrable on* Ω *and*

$$\int_{\partial K} N u \, d\sigma = \iint_{\mathring{K}} u' \, dx$$

for any compact Lipschitz domain $K \subset \Omega$.

(6) Du, *taken in the distribution sense, belongs to* $L^1_{\mathrm{loc}}(\Omega, \mathscr{A}_n)$.

If these equivalent conditions are fulfilled, then also $u' = Du$ *a.e. on* Ω.

THEOREM 7.3. *Let* u *be a* $\mathscr{A}_n$*-valued, uniformly* $(n-1)$*-integrable function in* $\mathbf{R}^n$, *which is absolutely continuous in the special Lipschitz domain* Ω *of* $\mathbf{R}^n \subset \mathscr{A}_n$. *Also, suppose that* $\operatorname{supp} u$ *is compact.*

$$\mu_{n-1}(\operatorname{supp} u \cap \bar{\Omega} \setminus \Omega) = 0 ,$$

u *is integrable on* $b\Omega$, *and that* Du *is integrable on* Ω.

Then

$$\int_{b\Omega} N u \, d\sigma = \iint_{\mathring{\Omega}} Du \, dx .$$

The next application is a refined version of the Pompeiu integral representation formula for $\mathscr{A}_n$-valued functions ([Mo], [Te]). To this effect, we shall call a locally $(n-1)$-integrable function u *mean-continuous at* $a \in \Omega$ if

$$\lim_{Q \downarrow a} \frac{1}{\mu_{n-1}(Q)} \int_{\partial Q} |u(x) - u(a)| \, d\sigma = 0 .$$

Also, let ω_n stand for the area of the unit sphere in $\mathbf{R}^n$.

THEOREM 7.4. *Let* Ω *be a compact Lipschitz domain in* $\mathbf{R}^n \subset \mathcal{A}_n$ *and let* u *be a* $\mathcal{A}_n$*-valued, uniformly* $(n-1)$*-locally integrable function on* $\mathbf{R}^n$, *which is absolutely continuous on* Ω *and mean-continuous almost everywhere on* Ω. *Then, at almost every point* $a \in \mathring{\Omega}$, *we have*

$$u(a) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{a-x}{|a-x|^n} N(x) u(x) d\sigma(x) + \frac{1}{\omega_n} \iint_{\Omega} \frac{x-a}{|x-a|^n} (Du)(x) dx .$$

This extends the results in [Te], [Mo], [Bo], [BDS], [HL]. Moreover, a similar result is valid for the Martinelli-Bochner integral representation formula (cf. [HL]).

THEOREM 7.5. *Assume that* Ω *is an open subset of* $\mathbf{R}^n \subset \mathcal{A}_n$. *Let* u *be a locally integrable,* $\mathcal{A}_n$*-valued function which is also locally* $(n-1)$*-integrable on* Ω. *Let* $(C_\nu)_\nu$ *be an at most countable collection of closed subsets of* Ω *such that each pair* (u, C_ν) *satisfies one of the conditions* (α)-(γ) *stated in* §6. *Set* $A := \cup_\nu C_\nu$ *and also let* f *be a locally integrable* $\mathcal{A}_n$*-valued function on* Ω.

Assume that at least one of the following conditions holds:

(1) A *is closed,* u *is integrally continuous on* $\Omega \backslash A$ *and* $Du = f$ *in the distribution sense on* $\Omega \backslash A$;

(2) u *is Clifford differentiable at each point of* $\Omega \backslash A$ *and* $u'(x) = f(x)$ *for any* $x \in \Omega \backslash A$.

Then $Du = f$ *in the distribution sense on* Ω.

Note that, for $f = 0$, Theorem 7.3 gives sufficient conditions for u to be monogenic. These are substantially weaker than the ones presented in the literature (cf. e.g. [BDS]).

In our final application we briefly explain how the above theorem extends to more general linear first order differential operators. In doing so, it is convenient to slightly alter the definition of uniform locally $(n-1)$-integrability, and replace (4.1) by

$$\int_C |u| \, d\sigma < \varepsilon .$$

With this modification, the uniform locally $(n-1)$-integrability condition becomes invariant under multiplication with locally bounded functions.

Also, a locally $(n-1)$-integrable function will be called *locally integrally bounded* in Ω, if for any $K \in \mathrm{comp}(\Omega)$ there exist $\theta, \kappa > 0$ such that for any Lipschitz $(n-1)$-dimensional submanifold C of $\mathbf{R}^n$, $C \subseteq K$, with $\mu_{n-1}(C) < \theta$ we have

$$\int_C |u|\, d\sigma < \kappa .$$

Consider now a linear, first order, differential operator

$$P = a_0(x) + \sum_{j=1}^{n} a_j(x)\partial_j ,$$

where the $\mathscr{A}_n$-valued functions $a_1, \ldots, a_n$ are locally Lipschitz continuous on Ω, and a_0 is a locally essentially bounded function on Ω. Let P^* stand for the formal transpose of P, i.e.

$$P^* = \left(a_0(x) - \sum_{j=1}^{n} (\partial_j a_j)(x)\right) + \sum_{j=1}^{n} a_j(x)\partial_j .$$

Also, for any $\xi \in \mathbf{R}^n$, the symbol of P is defined by $\sigma_P(\xi) := \sum_{j=1}^{n} \xi_j e_j a_j$. Recall that for two $\mathscr{A}_n$-valued, locally integrable functions u and f on Ω we have that $Pu = \mathrm{f}$ in the distribution sense, if

$$\iint_\Omega P^*(\psi)u = \iint_\Omega \psi f$$

for any real-valued test function ψ on Ω.

Let u be a locally $(n-1)$-integrable function on Ω. We shall say that u is *P-differentiable at* $x \in \Omega$ provided that the limit

$$Pu(x) := \lim_{Q \downarrow x} \frac{1}{\lambda_n(Q)} \left\{ \iint_Q P^*(1)u + \int_{\partial Q} \sigma_P(N)\, u\, d\sigma \right\}$$

exists in $\mathscr{A}_n$. Proceeding as in Theorem 6.2, one can readily see that if u is actually differentiable at $x \in \Omega$, and if

$$\lim_{Q \downarrow x} \frac{1}{\lambda_n(Q)} \iint_Q |a_0(y) - a_0(x)|\, dy = 0 ,$$

then u is P-differentiable at x and $Pu(x) = a_0(x)u(x) + \sum_{j=1}^{n} a_j(x)\partial_j u(x)$.

The following result is an extension of Theorem 3.1.10 in [Hö].

THEOREM 7.6. *With the above definitions, consider* u, f *two locally integrable* $\mathscr{A}_n$*-valued functions on* Ω, *and let* $(C_\nu)_\nu$ *be an at most countable collection of closed subsets of* Ω. *Assume that* u *is also locally* *integrally bounded. Suppose that at least one of the following conditions holds:*

(1) for each ν, *the pair* (u, C_ν) *satisfies the condition* (α)*;*

(2) $a_0 \equiv 0$ *and for each* ν, *the pair* (u, C_ν) *satisfies one of the conditions* $(\alpha) - (\gamma)$.

Finally, set $A := \cup_\nu C_\nu$ *and assume that* u *is P-differentiable at each point of* $\Omega \setminus A$ *and that* $Pu(x) = f(x)$ *for any* $x \in \Omega \setminus A$. *Then* $Pu = f$ *in the distribution sense on* Ω.

Let us finally note that, due to the non-commutativity of the Clifford algebra $\mathcal{A}_n$ for $n \geqslant 3$, the results presented in this section are not in the most general form. For instance, one could consider the Clifford differentiation operator defined for *ordered pairs* of $\mathcal{A}_n$-valued functions (u, v) by

$$(u, v)' := \lim_{Q \downarrow a} \frac{1}{\lambda_n(Q)} \int_{\partial Q} u N v \, d\sigma ,$$

for which all our techniques apply as well (cf. also [He1, 2]). However, we leave the details of this matter to the interested reader.

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