1. INTEGRAL AND ABSOLUTE CONTINUITY AND THE LOCAL PROBLEM

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1. Integral and absolute continuity and the local problem

DEFINITION 1.1. *A bounded subset* Ω *of* $\mathbf{R}^n$ *is called a Lipschitz domain if for any* $a \in \Omega \setminus \mathring{\Omega}$, *there exists an open neighborhood* U *of* a *in* $\mathbf{R}^n$, *a coordinate system (isometric to the canonical one)* $(x', x_n) = ((x_1, \ldots, x_{n-1}), x_n)$, *and a Lipschitz continuous function* $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ *such that*

$$\Omega \cap U = \{(x', x_n); \varphi(x') \leqslant x_n\} \cap U .$$

Also, if the new coordinates are actually obtained by permuting the canonical ones, then Ω *is called a simple Lipschitz domain.*

Note that, the *border* of the domain Ω, $b\Omega := \Omega \setminus \mathring{\Omega}$, is either the empty set or a $(n-1)$-dimensional Lipschitz submanifold of $\mathbf{R}^n$ (assumed with the standard induced orientation).

Let now Ω be a Lipschitz domain in $\mathbf{R}^n$ and ω an open set in $\mathbf{R}^{n-1}$. A locally bi-Lipschitz mapping $\varphi : \omega \to \Omega$ is called *Lipschitz embedding* provided φ maps ω homeomorphically onto $\varphi(\omega)$. Furthermore, if S is a topological space, $h : S \times \omega \to \Omega$ is called a *continuous family of Lipschitz embeddings* if $h_s := h(s, \cdot)$ is a Lipschitz embedding for each fixed $s \in S$, and if the mappings

$$S \ni s \mapsto \frac{\partial h_s}{\partial x_i} \in L^\infty(\omega, \mathrm{loc}), \quad i = 1, \ldots, n-1 , \tag{1.1}$$

are continuous. Here $L^\infty(\omega,\mathrm{loc})$ is endowed with the usual (Fréchet) topology given by uniform convergence on compact subsets of ω. Throughout this paper S will actually always be a locally closed subspace of some $\mathbf{R}^k$.

Let $L^1(\Omega, \mathrm{loc})$ stand for the vector space of differential forms with locally integrable coefficients on Ω. We consider this space endowed with the usual (locally convex) topology.

DEFINITION 1.2. *A complex-valued* $(n-1)$*-form* u *defined on* Ω *is called integrally continuous if:*

(1) the form u *is locally* $(n-1)$*-integrable, i.e.* $\varphi^* u$ *is locally integrable on* ω *for any Lipschitz embedding* $\varphi : \omega \to \Omega$;

(2) the mapping $S \ni s \mapsto h_s^* u \in L^1(\omega, \mathrm{loc})$ *is continuous, for any continuous family of Lipschitz embeddings* $h : S \times \omega \to \Omega$.

EXAMPLES. Let $u = \sum_{i=1}^n (-1)^{i-1} u_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ be a $(n-1)$-form on Ω where, as usual, the symbol under the "hat" is omitted in the product.

(1) If for $1 \leqslant i \leqslant n$ the functions $u_i|_C$ are μ_{n-1}-integrable for any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$, then u is locally $(n-1)$-integrable. In particular, this is the case if u_i are locally bounded.

(2) Suppose that there exists a set $A \subset \Omega$ of zero $(n-1)$-dimensional Hausdorff measure such that $u_i|_{(\Omega \setminus A)}$ is continuous (in the induced topology) and $u_i|_{C \cap (\Omega \setminus A)}$ is μ_{n-1} integrable for any $1 \leqslant i \leqslant n$ and any compact set $C \subset \Omega$ having $\mu_{n-1}(C) < +\infty$. Then u is integrally continuous as well.

(3) For $n = 1$, a $(n-1)$-form u is a function and, in this case, the form u is integrally continuous if and only if the function u is continuous.

Recall the usual exterior derivative operator d. The main result of this section is the following.

THEOREM 1.3. *Consider a Lipschitz domain* Ω *in* $\mathbf{R}^n$. *Let* u *be an integrally continuous* $(n-1)$*-form on* Ω *and let* f *be a locally integrable n-form on* Ω. *The following are equivalent.*

(1) For any compact Lipschitz domain $K \subseteq \Omega$ *we have* $\int_{\partial K} u = \iint_K f$.

(2) For any rectangle $Q \in \mathscr{R}(\Omega)$ *we have* $\int_{\partial Q} u = \iint_Q f$.

(3) $du = f$ *in the distribution sense on* $\overset{\circ}{\Omega}$.

Before we proceed with the proof of this theorem, we shall prove a lemma. To state it, we need some more notation. Let χ be a positive, smooth, function supported in the closed unit ball in $\mathbf{R}^n$ and normalized such that $\iint_{\mathbf{R}^n} \chi \, dx = 1$. For $\varepsilon > 0$, set $\Omega_\varepsilon := \{x \in \mathbf{R}^n; \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ and, for any $\Phi \in L^1(\overset{\circ}{\Omega}, \mathrm{loc})$, set

$$\Phi_\varepsilon(x) := \iint_{\mathbf{R}^n} \Phi(x - \varepsilon y)\, \chi(y)\, dy, \quad x \in \Omega_\varepsilon .$$

It is a well-known fact that $\Phi_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and that $\Phi_\varepsilon \to \Phi$ in $L^1(\overset{\circ}{\Omega}, \mathrm{loc})$ as ε tends to zero. For a locally integrable form u on Ω, u_ε is defined componentwise.

LEMMA 1.4. *Let* Ω *be a Lipschitz domain in* $\mathbf{R}^n$ *and let* u *be an integrally continuous* $(n-1)$*-form on* Ω. *Then:*

(1) $u \in L^1(\Omega, \mathrm{loc})$;

(2) $\varphi^* u_\varepsilon \to \varphi^* u$ *in* $L^1(\omega, \mathrm{loc})$ *as* ε *approaches zero, for any Lipschitz embedding* $\varphi : \omega \to \overset{\circ}{\Omega}$.

Proof. For each sufficiently small $\varepsilon > 0$, fixed for the moment, consider the continuous family of Lipschitz embeddings

$$h: \Omega_\varepsilon \times \omega_\varepsilon \ni (x, t) \mapsto (x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n),$$

where Ω_ε, defined above, stands for the space of parameters and ω_ε stands for a suitably small, open neighborhood of the cube $[-\varepsilon, \varepsilon]^{n-1}$. Obviously, $(h_x^* u)(t) = \Phi(x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \cdots \wedge dt_{n-1}$, for some function Φ. Since u is integrally continuous, the function

$$\Phi^\varepsilon(x) := \varepsilon^{1-n} \int_{[-\varepsilon, \varepsilon]^{n-1}} \Phi(x_1 + t_1, \dots, x_{n-1} + t_{n-1}, x_n) dt_1 \wedge \cdots \wedge dt_{n-1}$$

is continuous on Ω_ε. For any small, fixed x_n, the Lebesgue differentiation theorem yields that $\Phi^\varepsilon(\cdot, x_n) \to \Phi(\cdot, x_n)$, as $\varepsilon \to 0$, almost everywhere with respect to the $(n-1)$-dimensional Lebesgue measure on $\{x' \in \mathbf{R}^{n-1}; (x', x_n) \in \mathring{\Omega}\}$. Using Fubini's theorem we infer that $\Phi^\varepsilon \to \Phi$, as $\varepsilon \to 0$, almost everywhere on Ω. Thus, Φ is λ_n-measurable.

Next, let $Q = Q' \times Q_n$ be a rectangle in $\mathbf{R}^{n-1} \times \mathbf{R}$ which is contained in $\mathring{\Omega}$, and consider the continuous family of Lipschitz embeddings

$$k: Q_n \times Q' \ni (x_n, x') \mapsto (x', x_n) \in \Omega .$$

Hence, $k_{x_n}^* u = \Phi(\cdot, x_n) dx_1 \wedge \cdots \wedge dx_{n-1}$. As u is integrally continuous, the mapping

$$Q_n \ni x_n \mapsto \int_{Q'} |\Phi(x', x_n)| dx_1 \wedge \cdots \wedge dx_{n-1}$$

is continuous. In particular, the iterated integral

$$\int_{Q_n} \int_{Q'} |\Phi(x', x_n)| dx' \wedge dx_n$$

is finite. By Fubini's theorem, it follows that Φ is integrable on Q.

Now, if $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ the above reasoning gives that $u_1 = \Phi$ is integrable on Q. Likewise, $u_2, \dots, u_n$ are integrable on Q, and u is thus locally integrable on $\mathring{\Omega}$.

To conclude the proof of (1) it suffices to show that any $a \in \Omega \setminus \mathring{\Omega}$ has a compact neighborhood K in $\mathbf{R}^n$ such that u is integrable on $K \cap \Omega$. To see this, there is no loss of generality assuming that K is so that

$$K \cap \Omega = \{(x', x_n); x' \in Q', \varphi(x') \leqslant x_n \leqslant \varphi(x') + \varepsilon\},$$

where Q' is a rectangle in $\mathbf{R}^{n-1}$, $\varphi:\mathbf{R}^{n-1}\to\mathbf{R}$ is a Lipschitz function, and $\varepsilon>0$ is some fixed, sufficiently small number. This time we take the continuous family of Lipschitz embeddings

$$h':[0,\varepsilon]\times Q'\ni(s,x')\mapsto(x',\varphi(x')+s)\in\Omega$$

and proceed as before. Hence, (1) follows.

To see (2), let $u=\sum_{i=1}^{n}(-1)^{i-1}u_i dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_n$ in Ω, so that we have $\varphi^*u=(\sum_i(u_i\circ\varphi)\Phi_i)dt_1\wedge\cdots\wedge dt_{n-1}$, where Φ_i are measurable functions, locally (essentially) bounded on ω. Similarly, $\varphi^*u_\varepsilon=(\sum_i((u_i)_\varepsilon\circ\varphi)\Phi_i)dt_1\wedge\cdots\wedge dt_{n-1}$.

Given a compact subset C of ω, we consider the continuous family of Lipschitz embeddings $(s,t)\mapsto\varphi(t)-s$, where t lies in an open neighborhood of C and s lies in a small open ball centered at the origin of $\mathbf{R}^n$. Let also $\theta>0$ be an arbitrary, fixed number. By the integral continuity of u, there exists $\delta>0$ such that

$$(1.2)\qquad \int_C\left|\sum_{i=1}^{n}u_i(\varphi(t)-\varepsilon y)\Phi_i(t)-\sum_{i=1}^{n}u_i(\varphi(t))\Phi_i(t)\right|\chi(y)dt<\theta$$

for all $|y|\leqslant 1$ and $0<\varepsilon<\delta$. Since, by (1), the functions u_i are locally integrable on $\mathring{\Omega}$, the function

$$\varepsilon^{-n}\sum_i u_i(y)\Phi_i(t)\chi\left(\frac{\varphi(t)-y}{\varepsilon}\right),\quad \varepsilon>0,$$

is locally integrable on $\omega\times\mathring{\Omega}$. Integrating (1.2) against dy over the closed unit ball in $\mathbf{R}^n$ and then changing the order of integration, we obtain

$$\int_C|\varphi^*u_\varepsilon-\varphi^*u|\,d\mu_{n-1}\leqslant c_n\theta,$$

for some $c_n>0$ depending only on n. Since $\theta>0$ was arbitrary, the proof of the lemma is therefore complete. □

Proof of Theorem 1.3. Obviously (1) implies (2). Next, assume that (2) holds and let Q be an arbitrary rectangle in $\mathbf{R}^n$ contained in $\mathring{\Omega}$. It is then straightforward to see that, for a sufficiently small $\varepsilon>0$,

$$\int_{\partial Q}u_\varepsilon=\iint_Q f_\varepsilon.$$

Since u_ε is smooth, the standard form of Stokes formula gives that $\iint_Q du_\varepsilon = \iint_Q f_\varepsilon$. As Q was arbitrarily chosen, we see that $du_\varepsilon = f_\varepsilon$ on $\mathring{\Omega}_\varepsilon$ and, hence, by letting ε go to zero, $du = f$ in the distribution sense on $\mathring{\Omega}$. Thus, (2) $\Rightarrow$ (3).

Finally, we consider the implication (3) $\Rightarrow$ (1). Using a smooth partition of unity, it is not difficult to see that matters can be reduced to verifying (1.1) in the following cases:

(i) the support of u is included in the interior of K;

(ii) the domain K has the form

$$\{(x', x_n) \in [0, 1]^{n-1} \times [0, 1]; x_n \leqslant \varphi(x')\}, \tag{1.3}$$

for some Lipschitz function $\varphi: \mathbf{R}^{n-1} \to (0, 1)$.

We present the proof in the second case, as the proof the first case goes along the same lines and is somewhat simpler. Let us first note that, if K_ε is as in (1.3) except that φ has been replaced by $\varepsilon\varphi$, with $0 < \varepsilon < 1$, on account of the integral continuity of u we have

$$\int_{\partial K} u = \lim_{\varepsilon \to 1} \int_{\partial K_\varepsilon} u .$$

Hence, it suffices to prove the statement with K_ε in place of K or, in other words, assuming that the compact domain K from (1.2) is actually contained in $\mathring{\Omega}$. Furthermore, since by (1) $du_\varepsilon = f_\varepsilon$ on Ω_ε for all $\varepsilon > 0$, and since

$$\int_{\partial K} u_\varepsilon \to \int_{\partial K} u, \quad \iint_K f_\varepsilon \to \iint_K f$$

as $\varepsilon \to 0$ (the first convergence utilizes the integral continuity of u), there is no loss in generality if we assume that u and f are smooth forms in a neighborhood of K.

Consider now the bi-Lipschitz homeomorphism

$$h: [0, 1]^n \ni (x', x_n) \mapsto (x', x_n \varphi(x')) \in K .$$

From the change of variable formula ([Fe3], Theorem 3.2.3, p. 243) we have

$$\int_{\partial K} u = \int_{\partial [0,1]^n} h^* u .$$

Also, a routine calculation shows that

$$(h^*u)(x', x_n) = v(x', x_n) + \left(\sum_{i=1}^{n-1} w_i(x', x_n) \partial_i \varphi(x') \right) dx_1 \wedge \cdots \wedge dx_{n-1} ,$$

where the coefficients of the $(n-1)$-form v as well as $(w_i)_i$ are Lipschitz functions. Clearly, the usual Stokes formula on $[0,1]^n$ holds for v whereas, for $1 \leqslant i \leqslant n-1$,

$$\int_{[0,1]^{n-1}} (w_i(x',1) - w_i(x',0))\partial_i\varphi(x')dx'$$

$$= (-1)^{n-1}\int_0^1\int_{[0,1]^{n-1}} \frac{\partial w_i(x',x_n)}{\partial x_n}\partial_i\varphi(x')dx' \wedge dx_n .$$

Consequently, the Stokes formula holds for h^*u on $[0,1]^n$, so that

$$\int_{\partial K} u = \int_{\partial[0,1]^n} h^*u = \iint_{[0,1]^n} d(h^*u) = \iint_{[0,1]^n} h^*(du)$$

$$= \iint_{[0,1]^n} h^*f = \iint_K f$$

and the proof is complete. □

DEFINITION 1.5. *Let* Ω *be a Lipschitz domain in* $\mathbf{R}^n$. *An integrally continuous* $(n-1)$*-form* u *on* Ω *is called absolutely continuous on* Ω *if* $d(u\,|\,_{\mathring{\Omega}})$, *taken in the distribution sense, is integrable on* $\mathring{K}$ *for any compact subset* K *of* Ω.

Note that if $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ and u_i are, for instance, locally Lipschitz on Ω, then u is absolutely continuous on Ω.

A simple consequence of Theorem 1.3 and of the above definition is the next.

THEOREM 1.6. *If* K *is a compact Lipschitz domain in* $\mathbf{R}^n$ *and* u *is an absolutely continuous* $(n-1)$*-form on* K, *then*

$$\int_{\partial K} u = \iint_{\mathring{K}} du .$$

2. CHARACTERIZATIONS OF THE PLURIDIMENSIONAL ABSOLUTE CONTINUITY

Theorem 1.3 suggests the possibility of characterizing pluridimensional absolute continuity of $(n-1)$-forms in a way similar to Lebesgue's definition of absolute continuity of functions on the real line (i.e. without involving the exterior derivative operator). This is made precise in the following theorem.