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6. Tests for the equalities $du = f$ and $\bar{\partial} u = f$ in the weak sense

Let $\Omega \subseteq \mathbf{R}^n$ be a fixed open set in $\mathbf{R}^n$. In this section we let $\mathscr{R}(\Omega)$ denote the collection of all rectangles Q contained in Ω and having $p(Q) \leqslant c_0$, for a fixed real number $1 < c_0 < +\infty$.

First, we shall give a coordinate free definition of the exterior differentiation operation. This builds on the classical work of Pompeiu [Po2] for the case $n = 2$ (cf. also the results in §7).

DEFINITION 6.1. *A $(n-1)$-form u which is locally $(n-1)$-integrable on Ω is said to be exteriorly differentiable at $a \in \Omega$ if the limit*

$$c := \lim_{Q \downarrow a} \frac{1}{\lambda_n(Q)} \int_{\partial Q} u$$

exists in $\mathbf{C}$. More specifically, we assume that there exists a complex number c so that, for any $\varepsilon > 0$, there exists an open neighborhood $U \subseteq \Omega$ of a such that

$$\left| \int_{\partial Q} u - c\lambda_n(Q) \right| < \varepsilon \lambda_n(Q) ,$$

for all $Q \in \mathscr{R}(\Omega)$, $Q \subset U$.

We then set $u'(a) := c$ and $du_a := c\pi_1 \wedge \cdots \wedge \pi_n$, where $\pi_1, \ldots, \pi_n$ are the canonical coordinate projections of $\mathbf{R}^n$.

For $n = 1$, u' becomes the usual derivative of the function u. Our next theorem collects several exterior differentiability criteria for $(n-1)$-forms.

THEOREM 6.2. *Let $u = \sum_{i=1}^n (-1)^{i-1} u_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ be a locally $(n-1)$-integrable form on Ω.*

(1) If the function $u_i, i = 1, \ldots, n$ are differentiable at $a \in \Omega$, then u is exteriorly differentiable at a and

$$u'(a) = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i}(a) .$$

(2) If u satisfies one of the equivalent conditions in Theorem 1.3, then u is exteriorly differentiable at almost every point of Ω and $u' \in L^1(\Omega, \mathrm{loc})$. In particular, this is the case if u is absolutely continuous or integrally Lipschitz on Ω.

Proof. By hypotheses, there exist some numbers $c_{ij} \in \mathbf{R}$ and some functions ξ_i which are continuous and vanish at a, such that

$$u_i(x) = u_i(a) + \sum_{j=1}^{n} c_{ij}(x_j - a_j) + \xi_i(x)\,\|x - a\|, \quad i = 1, 2, \ldots, n\,.$$

A straightforward computation then yields $u'(a) = \sum_{i=1}^{n} c_{ii}$.

The second part of the conclusion follows directly from Theorem 1.3 and Lebesgue's differentiation theorem. □

Now we consider a p-form $u = \sum'_{|I|=p} u_I dx^I$ on $\Omega, 0 \leqslant p \leqslant n-1$. Here $\sum'$ indicates that the sum is performed over the set of all strictly increasing multi-indices I of length p, i.e. all ordered p-tuples of the form $I = (i_1, \ldots, i_p)$, with $1 \leqslant i_1 < \cdots < i_p \leqslant n$. Also, dx^I stands for $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ if $I = (i_1, \ldots, i_p)$. For each strictly increasing multi-index J of length $p+1$ we introduce the $(n-1)$-form

$$u^J := \sum_{i=1}^{n} (-1)^{i-1} \left(\sum_{|I|=p}' \varepsilon_J^{iI} u_I \right) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\,.$$

Here $\varepsilon_J^{iI} = 0$ unless $\{i\} \cup I = J$, in which case ε_J^{iI} is the sign of the permutation taking iI, the concatenation of $\{i\}$ and I, onto J.

The forms u^J will be called the *$(n-1)$-forms associated to u*. Since, clearly, the application

$$u \mapsto \{u^J; |J| = p+1\}$$

is one-to-one, we can represent a given differential form either by its coefficients, or by the $(n-1)$-forms associated to it. In fact, for $p = n-1$, the functions $u^J, |J| = n$, are precisely the coefficients of the form u. Furthermore, one can easily check that u is locally integrable if and only each of its associated $(n-1)$-forms is locally integrable.

It is natural to use the associated $(n-1)$-forms to extend the concepts already defined for $p = n-1$ to the general case of p-forms, $p \leqslant n-1$. More specifically, a p-form is called *locally $(n-1)$-integrable, exteriorly differentiable at a*, etc, if all its associated $(n-1)$-forms have that particular property. In the case when u^J's are exteriorly differentiable at $a \in \Omega$, we also set

$$du_a := \sum_{|J|=p+1}' (u^J)'(a)\,\pi^J\,,$$

where $\pi^J := \pi_{j_1} \wedge \cdots \wedge \pi_{j_{p+1}}$ if $J = (j_1, \ldots, j_{p+1})$.

Suppose now that $u = \sum_{i=1}^{n}(-1)^{i-1}u_i\,dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ is a $(n-1)$-form on Ω. For $0 \leqslant r \leqslant 1, x \in \Omega$, and $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$, we set

$$\omega_x(u, \varepsilon) := \sup_{y \neq z \in B_\varepsilon(x)} \frac{\| u(y) - u(z) \|}{\| y - z \|^r} .$$

where $B_\varepsilon(x) \subset \mathbf{R}^n$ is the ball of radius ε centered at x and $u(x)$ is identified with the point $(u_1(x), \ldots, u_n(x))$ of $\mathbf{R}^n$, etc.

For a $(n-1)$-form u on Ω and a subset $C \subset \Omega$, we consider the following conditions:

Condition (α). $\mu_{n-1}(C) = 0$, u is locally $(n-1)$-integrable on Ω and uniformly locally $(n-1)$-integrable on a neighborhood of C.

Condition (β). There exists some $0 < r \leqslant 1$ such that $\mu_{n+r-1}(C) = 0$, u is uniformly locally $(n-1)$-integrable on Ω and has the property that

$$\omega_x(u, \varepsilon) = O(1), \quad \text{as } \varepsilon \to 0 , \tag{6.1}$$

at each point x of Ω outside some closed, μ_{n-1}-negligible set $A \subset \Omega$.

Condition (γ). There exists some $0 \leqslant r < 1$ such that $\mu_{n+r-1}(C) < +\infty$, u is uniformly locally $(n-1)$-integrable on Ω and has the property that

$$\omega_x(u, \varepsilon) = o(1), \quad \text{as } \varepsilon \to 0 , \tag{6.2}$$

at each point x of Ω outside some closed, μ_{n-1}-negligible set $A \subset \Omega$.

The main results of this section are the following.

THEOREM 6.3. *Consider a complex-valued, locally $(n-1)$-integrable p-form u on Ω. Let $(C_\nu)_\nu$ be an at most countable collection of closed subsets of Ω such that, for each ν and each associated $(n-1)$-form u^J of u, the pair (u^J, C_ν) satisfies one of the conditions (α)-(γ) stated above. Furthermore, assume that for any multiindex J*

$$\limsup_{Q \downarrow x} \frac{1}{\lambda_n(Q)} \left| \int_{\partial Q} u^J \right| < +\infty \tag{6.3}$$

at any $x \in \Omega \setminus (\cup_\nu C_\nu)$.

Then, for each J, the restriction of u^J to any relatively compact open subdomain of Ω is integrally Lipschitz. In particular, u is exteriorly differentiable almost everywhere on Ω.

THEOREM 6.4. *Let u be a complex-valued locally integrable p-form which is locally $(n-1)$-integrable on Ω and let $(C_\nu)_\nu$ be a as in*

Theorem 6.3. Also, set $A := \cup_\nu C_\nu$ *and consider a complex-valued* $(p+1)$*-form* f *in* $L^1(\Omega, \mathrm{loc})$. *Furthermore, assume that at least one of the following conditions is fulfilled:*

(1) A *is closed,* u *is integrally continuous on* $\Omega \setminus A$ *and* $du = f$ *in the distribution sense on* $\Omega \setminus A$*;*

(2) u *is exteriorly differentiable on* $\Omega \setminus A$ *and* $du_x = f_x$ *at each point* $x \in \Omega \setminus A$.

Then $du = f$ *in the distribution sense on* Ω.

REMARK 6.5. For integrally continuous forms u such that the limit in (6.3) vanishes for each J, Theorem 6.3 gives sufficient conditions for the equality $du = 0$ to hold in the distribution sense on Ω.

Moreover, in the case $f = 0$, Theorem 6.4 furnishes tests for a p-form to be closed, too. Theorem 6.3 also gives absolute continuity criteria for integrally continuous forms. In turn, these can be used to further improve the main results of §4 and §5.

The proofs of these theorems will be accomplished in a series of lemmas.

LEMMA 6.6. *Let* u *be a locally* $(n-1)$*-integrable* $(n-1)$*-form on* Ω, f *a locally integrable n-form on* Ω, *and let*

$$\varphi(Q) := \int_{\partial Q} u - \iint_Q f, \tag{6.4}$$

for $Q \in \mathscr{R}(\Omega)$. *Also, let* C *be a closed subset of* Ω. *If the pair* (u, C) *fulfills one of the conditions* (α)-(γ) *stated above, then the set* C *is* $(\varphi, 0)$*-negligible.*

Proof. If (α) is the fulfilled condition, then the statement follows from an obvious variant of Lemma 4.2, (3). To complete the proof in the remaining cases, let us consider $0 \leqslant r \leqslant 1$ such that C has finite $(n+r-1)$-dimensional Hausdorff measure. Also, let $A \subset \Omega$ with $\mu_{n-1}(A) = 0$ be the exceptional set appearing in the statement of the conditions (β) and (γ). Finally, we fix a rectangle $Q \in \mathscr{R}(\Omega)$ and two small numbers $\varepsilon, \delta > 0$.

Consider now two paved sets $P, R \subseteq Q$ such that $\mathring{Q} \cap C \subset \mathring{P}$ and $\mathring{Q} \cap A \subset \mathring{R}$. Without any loss of generality we can assume that $0 < \varepsilon < \operatorname{dist}(A, \partial P)$ and that $\mu_{n-1}(\partial R) < \delta$. We can also assume that there exist finitely many cubes $R_1, \ldots, R_m$ with diameters inferior to ε so that $(R_\nu)_{\nu=1}^m$ is a subdivision of $P \setminus \mathring{R}$ and such that

$$\sum_{\nu=1}^m \operatorname{diam}(R_\nu)^{n+r-1} \leqslant \mu_{n+r-1}(C) + \varepsilon .$$

Next, let $(Q_i)_{i \in I}$ be a subdivision of Q such that $(Q_i)_{i \in I_1} = (R_\nu)_{\nu=1}^m$ for some $I_1 \subseteq I$ and that, for some $I_2 \subseteq I$, $(Q_i)_{i \in I_2}$ is a subdivision of R. We set $J := I_1 \cup I_2$. As a consequence, $Q_i \cap A = \varnothing$ for each $i \notin J$. Also,

$$\sum_{i \in J} \varphi\,(Q_i) = \sum_{\nu=1}^m \int_{\partial R_\nu} u + \int_{\partial R} u - \iint_{P \cup R} f .$$

Going further, for $\nu = 1, \ldots, m$ we fix some points $x_\nu \in R_\nu$ and set

$$u(x_\nu) := \sum_{i=1}^n (-1)^{i-1} u_i(x_\nu)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots dx_n .$$

Then

$$\left| \int_{\partial R_\nu} u \right| = \left| \int_{\partial R_\nu} (u - u(x_\nu)) \right| \leqslant \sum_\sigma \left| \int_\sigma (u - u(x_\nu)) \right| ,$$

where the sum runs over the faces of R_ν. For instance, if σ is a face of R_ν on which $x_1 =$ constant, then

$$\left| \int_\sigma (u_1 - u_1(x_\nu))\, dx_2 \wedge \cdots \wedge dx_n \right| \leqslant \sup_{x \in \sigma} | u_1(x) - u_1(x_\nu) | \, \mu_{n-1}(\sigma) .$$

All in all, we get that

$$\left| \int_{\partial R_\nu} u \right| \leqslant c_n \omega_{x_\nu}(u, \varepsilon) \operatorname{diam}(R_\nu)^{n+r-1} ,$$

for some positive constant c_n depending solely on n. Adding up in ν we obtain

$$\sum_{\nu=1}^m \left| \int_{\partial R_\nu} u \right| \leqslant c_n(\mu_{n+r-1}(C) + \varepsilon) \max_{1 \leqslant \nu \leqslant m} \omega_{x_\nu}(u, \varepsilon) . \tag{6.5}$$

Now, given $\theta > 0$, there exist $\varepsilon_0, \delta_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, $0 < \delta < \delta_0$ we have $| \iint_{P \cup R} f | < \theta/3$. Also, if δ_0 is sufficiently small, from the uniform locally $(n-1)$-integrability of u we infer that $| \int_{\partial R} u | < \theta/3$.

At this point, we fix ε_0, δ_0 and, by (6.1) (or 6.2), respectively), conclude that $\omega_{x_\nu}(u, \varepsilon) = O(1)$ (or $o(1)$, respectively) as $\varepsilon \to 0$, uniformly in ν. Using this, (6.4) and the assumptions concerning the size of $\mu_{n+r-1}(C)$, we get $|\int_{\partial(P \setminus \mathring{R})} u| < \theta/3$, provided ε is small enough.

Summarizing, for ε and δ as above, we see that $|\sum_{i \in J} \varphi(Q_i)| < \theta$, and the conclusion follows. $\square$

LEMMA 6.7. *Let* u *be a* $(n-1)$*-form which is locally* $(n-1)$*-integrable on* Ω *and has real-valued coefficients, and let* $(C_\nu)_\nu$ *be an at most countable collection of closed subsets of* Ω *such that each pair* (u, C_ν) *satisfies one of the conditions* $(\alpha)-(\gamma)$. *Set* $A := \cup_\nu C_\nu$ *and let* f *be a locally integrable n-form on* Ω, *also having real-valued coefficients.*

If u *is exteriorly differentiable on* $\Omega \setminus A$ *and* $u'(x) \leqslant f(x)$ *for all* $x \in \Omega \setminus A$, *then*

$$\int_{\partial Q} u \leqslant \iint_Q f \tag{6.6}$$

for any $Q \in \mathscr{R}(\Omega)$.

Proof. Let us first assume that f is lower semi-continuous on Ω. We shall verify the condition (2) in Theorem 3.4 for the additive functions φ introduced in (6.3), and $t := \lambda_n$. To this effect, let us fix $a \in \Omega \setminus A$ and consider a nested sequence of rectangles $(Q_\nu)_\nu$ such that $\cap_\nu Q_\nu = \{a\}$. Since u is exteriorly differentiable at a and since f is lower semi-continuous it follows that

$$\liminf_\nu \frac{1}{\lambda_n(Q_\nu)} \int_{\partial Q_\nu} U = u'(a) \leqslant f(a) \leqslant \limsup_\nu \frac{1}{\lambda_\nu(Q_\nu)} \iint_{Q_\nu} f.$$

Consequently,

$$\liminf_\nu \frac{\varphi(Q_\nu)}{\lambda_n(Q_\nu)} \leqslant 0$$

and the conclusion is provided in this case by the equivalence (1) $\Leftrightarrow$ (2) in Theorem 3.4.

Finally, as

$$\iint_Q f = \inf \left\{ \iint_Q g ;\, g \text{ lower semi-continuous and } \geqslant f \right\},$$

the general case obviously reduces to the one just considered. $\square$

LEMMA 6.8. *Suppose that* u, f, A, *are as in the first part of Lemma 6.6. In addition, assume that at least one of the following two conditions holds:*

(1) A *is closed and* $\int_{\partial Q} u = \iint_Q f$ *for any* $Q \in \mathscr{R}(\Omega)$ *such that* $Q \cap A = \emptyset$;

(2) u *is exteriorly differentiable on* $\Omega \setminus A$ *and* $du_x = f_x$ *for all* $x \in \Omega \setminus A$.

Then

$$\int_{\partial Q} u = \iint_Q f$$

for any $Q \in \mathscr{R}(\Omega)$.

Proof. In the first case the assertion follows directly from Lemma 6.6 and Theorem 3.4. As for the second one, the conclusion is immediately seen from Lemma 6.6. □

LEMMA 6.9. *Consider* $f = \sum'_{|J|=p+1} f_J dx^J$ *a locally integrable* $(p+1)$*-form on* Ω, *and let* u *be a locally integrable* p*-form on* Ω. *Then* $du = f$ *in the distribution sense if and only if* $du^J = f_J dx_1 \wedge \cdots \wedge dx_n$ *in the distribution sense for any* J, $|J| = p + 1$.

Proof. For any smooth form v and for any $|J| = p+1$, a routine calculation shows that

$$dv^J = (dv)_J dx_1 \wedge \cdots \wedge dx_n .$$

The general case then follows from this observation and a standard regularization technique. □

Now we are ready to present the proofs of the main results of this section.

Proof of Theorem 6.3. The conclusions of the theorem are readily seen from Lemma 6.6, Theorem 3.5 and Theorem 6.2. □

Proof of Theorem 6.4. Using Lemma 6.8 one can reduce matters to $p = n-1$, in which case the theorem follows from Lemma 6.7 and Theorem 1.3. □

In the last part of this section we shall present similar results for the usual $\bar{\partial}$ operator acting on differential forms. Let $\Omega \subset \mathbf{C}^n$ be an open set, and let

$$u = \sum_{|I|=p}{}' \sum_{|K|=q}{}' u_{I,K} dz^I \wedge d\bar{z}^K$$

be a (p, q)-form on Ω, $0 \leqslant p \leqslant n$, $0 \leqslant q \leqslant n-1$. For any multi-indices I, J with $|I| = p$ and $|J| = q+1$, we set

$$u^{I,J} := (-1)^{n+p} \sum_{j=1}^{n} (-1)^{j-1} \left(\sum_{I,J}{}' \varepsilon_J^{jK} u_{I,K} \right) dz^{\{1,2,\dots,n\}} \wedge d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots d\bar{z}_n.$$

The forms $u^{I,J}$ are called the *$(n, n-1)$-forms associated to u*. The concepts of integral continuity, etc, are introduced for (p, q)-forms as in the real case. We have the following.

THEOREM 6.10. *Let u be a locally integrable, complex-valued form of type (p, q), which is also locally $(n-1)$-integrable on an open subset Ω of $\mathbf{C}^n$. Let $(C_\nu)_\nu$ be a sequence of closed subsets of Ω such that each pair $(u^{I,J}, C_\nu)$ satisfies one of the conditions (α)-(γ). Also, let $A = \cup_\nu C_\nu$ and let f be a locally integrable form of type $(p, q+1)$ on Ω.*

Assume that at least one of the following conditions is valid:

(1) A is closed, u is integrally continuous on $\Omega \backslash A$ and $\bar{\partial} u = f$ in the distribution sense on $\Omega \backslash A$;

(2) u is exteriorly differentiable on $\Omega \backslash A$ and $\bar{\partial} u_x = f_x$ at each point $x \in \Omega \backslash A$.

Then $\bar{\partial} u = f$ in the distribution sense on Ω.

The proof is completely similar to the proof of the Theorem 6.4, hence omitted.

REMARK 6.11. For $f = 0$ we obtain tests for a (p, q)-form to be $\bar{\partial}$-closed, and for $p = q = 0$ tests for a function u to be holomorphic. The latter are well-known and due to Pompeiu [Po1] in the case $n = 1$. Our theorem also extends the holomorphy tests of [BM] and [Shi] in the case $n \geqslant 2$. Note that for $n = 2$, $p = q = 0$, $A = \varnothing$ and $f = 0$, we obtain the classical Goursat lemma.

Before we conclude this section, let us note that Theorem 6.3 naturally extends to the several complex variable setting and that this can also be used to obtain holomorphy criteria (cf. also [L]).