

## 4. $SS^1$ -FIBRATIONS

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **41 (1995)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **09.08.2024**

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indeed by [GN<sub>1</sub>, Proposition 5.7], for odd  $p$ ,  $\Gamma$  is cyclic of order  $2p^2$ . The proof there also shows that  $2[\gamma_{1,1}]$  is of order  $p^2$  and that  $p[\gamma_{0,p}]$  is of order 2 in  $\Gamma$ , so  $[\gamma_{2,2+p^2}]$  generates  $\Gamma$ .

#### (D) THE PROJECTIVE PLANE

We saw that when  $X$  is aspherical and  $\chi(X) \neq 0$  then  $\Gamma = 0$  and so our first order invariants vanish. In the presence of non-trivial higher homotopy these invariants need not vanish, despite  $\chi(X) \neq 0$ , as demonstrated by the example of the real projective plane  $X = P^2$ .

Write  $G \equiv \pi_1(P^2) \cong \mathbf{Z}/2$ ; denote the generator of  $G$  by  $t$ . Give  $P^2$  the customary cell structure consisting of one cell in each of dimensions 0, 1, and 2. The universal cover  $\tilde{P}^2$  is naturally identified with  $S^2$  and the corresponding cellular chain complex is:

$$C_2(S^2) \xrightarrow{1+t^{-1}} C_1(S^2) \xrightarrow{t^{-1}-1} C_0(S^2).$$

Every element of  $\Gamma$  can be represented by a basepoint preserving homotopy  $F: P^2 \times I \rightarrow P^2$  with  $F_0 = F_1 = \text{id}_{P^2}$ . We have  $\tilde{F}_0 = \tilde{F}_1 = \text{id}_{S^2}$  because the basepoint is preserved. It is easy to verify that the corresponding chain homotopy  $\tilde{D}_*: C_*(S^2) \rightarrow C_*(S^2)$  is then zero on  $C_0(S^2)$  and takes  $\tilde{e}_1$  to  $\tilde{e}_2 m(1 - t^{-1})$  where  $m \in \mathbf{Z}$ . By elementary obstruction theory, there exists  $F \equiv F^{(m)}$  realizing any  $m \in \mathbf{Z}$ . In this case  $\text{trace}(\tilde{\partial} \otimes \tilde{D}) = (1 + t^{-1}) \otimes m(1 - t^{-1})$  which is homologous to the canonical form  $mt^{-1} \otimes tt^{-1} - mt^{-1} \otimes tt^{-2}$ . Since  $\chi(P^2) = 1 \neq 0$ , the Gottlieb group  $\eta_{\#}(\Gamma) \equiv \mathcal{G}(P^2) = 0$  and so the derivation  $\tilde{X}_1(P^2)$  is a homomorphism and need not be distinguished from its cohomology class  $\tilde{\chi}_1(P^2) \in H^1(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2))) \cong \text{Hom}(\Gamma, HH_1(\mathbf{Z}(\mathbf{Z}/2)))$ . It follows that

$$\tilde{\chi}_1(P^2) ([F^{(m)}]) = (m, -m) \in \mathbf{Z}/2 \oplus \mathbf{Z}/2 \cong HH_1(\mathbf{Z}(\mathbf{Z}/2)).$$

In particular, when  $m$  is odd  $\tilde{\chi}_1(P^2) ([F^{(m)}]) \neq 0$ . On the other hand, this shows  $\chi_1(P^2) = 0$ .

#### 4. $S^1$ -FIBRATIONS

In this section we investigate the first order Euler characteristic of the total space of an orientable Serre fibration with  $S^1$ -fiber.

Let  $S^1 \rightarrow X \xrightarrow{\pi} B$  be an orientable Serre fibration where  $B$  is a (not necessarily finite) connected CW complex and  $X$  has the homotopy type of a finite complex. By classical obstruction theory, fiber homotopy

equivalence classes of orientable  $S^1$ -fibrations over a CW complex  $B$  are classified by the integral cohomology group  $H^2(B; \mathbf{Z})$ . Given an element  $e \in H^2(B; \mathbf{Z}) \cong [B, \mathbf{CP}^\infty]$  one obtains a principal  $U(1)$ -bundle over  $B$  by pulling back, via a continuous map  $B \rightarrow \mathbf{CP}^\infty$  representing  $e$ , the  $U(1)$ -bundle associated to the canonical complex line bundle over the infinite dimensional complex projective space  $\mathbf{CP}^\infty$ . Thus we can assume, without loss of generality, that  $S^1 \rightarrow X \xrightarrow{\pi} B$  is a principal  $U(1)$ -bundle. In particular, there is a free  $U(1)$ -action on  $X$  which we will write as  $\Phi: X \times S^1 \rightarrow X$ . Let  $\tau \in \Gamma \equiv \pi_1(\mathcal{E}(X), 1)$  be the element represented by  $\Phi$  ( $\Phi = \Phi^\tau$  in the notation of §1). For any coefficient ring  $R$ , let  $\{r\} \in H_1(X; R)$  denote the image of  $\tau$  under the composite:

$$\Gamma \xrightarrow{\eta} \pi_1(X) \rightarrow H_1(X) \rightarrow H_1(X; R) .$$

Also, let  $e_R$  be the image of the element  $e \in H^2(B; \mathbf{Z})$  which classifies  $S^1 \rightarrow X \xrightarrow{\pi} B$  under the homomorphism  $H^2(B; \mathbf{Z}) \rightarrow H^2(B; R)$ .

LEMMA 4.1. *If  $\mathbf{F}$  is a field, then  $\{\tau\} \in H_1(X; \mathbf{F})$  is non-zero if and only if  $e_{\mathbf{F}} = 0$ .*

*Proof.* Consider the Gysin homology sequence for the fibration  $S^1 \rightarrow X \xrightarrow{\pi} B$ :

$$\cdots \rightarrow H_2(B; \mathbf{F}) \xrightarrow{e_{\mathbf{F}} \cap} H_0(B; \mathbf{F}) \xrightarrow{\theta_0} H_1(X; \mathbf{F}) \xrightarrow{\pi_*} H_1(B; \mathbf{F}) \rightarrow 0 .$$

Since  $H_2(B; \mathbf{F}) \xrightarrow{e_{\mathbf{F}} \cap} H_0(B; \mathbf{F}) \cong \mathbf{F}$  is just evaluation of the cohomology class  $e_{\mathbf{F}}$  on homology,  $\theta_0$  is non-zero if and only if  $e_{\mathbf{F}} = 0$ . Let  $v \in X$  be a basepoint and let  $\{\pi(v)\} \in H_0(B; \mathbf{F})$  be the generator determined by the inclusion of  $\pi(v)$  into  $B$ . The fact that  $\theta_0(\{v\}) = \{\tau\}$  follows from the naturality of the Gysin sequence homology sequence, by mapping the Gysin sequence of the trivial fibration  $S^1 \rightarrow S^1 \rightarrow \pi(v)$ , via the homomorphism induced by inclusion, into the Gysin sequence for  $S^1 \rightarrow X \xrightarrow{\pi} B$ .  $\square$

THEOREM 4.2. *Let  $\mathbf{F}$  be a field. If  $e_{\mathbf{F}} \neq 0$  then  $\chi_1(X; \mathbf{F})(\tau) = 0$ . If  $e_{\mathbf{F}} = 0$  then  $H_*(B; \mathbf{F})$  is finite dimensional over  $\mathbf{F}$  and  $\chi_1(X; \mathbf{F})(\tau) = -\chi(B; \mathbf{F})\{\tau\}$  where  $\chi(B; \mathbf{F}) = \sum_{i \geq 0} (-1)^i \dim_{\mathbf{F}} H_i(B; \mathbf{F})$ .*

*Proof.* In this proof, all homology and cohomology groups will have coefficients in the field  $\mathbf{F}$ . Since  $B$  is the orbit space of the  $U(1)$ -action on  $X$  given by  $\Phi$ , there is a commutative square:

$$\begin{array}{ccc} X \times S^1 & \xrightarrow{\Phi} & X \\ \pi \times \text{id} \downarrow & & \pi \downarrow \\ B \times S^1 & \xrightarrow{p} & B \end{array}$$

where  $p: B \times S^1 \rightarrow B$  is projection. This square induces a commutative ladder mapping the Gysin homology sequence of  $S^1 \rightarrow X \times S^1 \xrightarrow{\pi \times \text{id}} B \times S^1$  to the Gysin homology sequence of  $S^1 \rightarrow X \xrightarrow{\pi} B$ :

$$\begin{array}{ccccccc}
 H_i(B \times S^1) & \xrightarrow{\theta'} & H_{i+1}(X \times S^1) & \xrightarrow{(\pi \times \text{id})^*} & H_{i+1}(B \times S^1) & \rightarrow & H_{i-1}(B \times S^1) \\
 p_* \downarrow & & \Phi_* \downarrow & & p_* \downarrow & & p_* \downarrow \\
 H_i(B) & \xrightarrow{\theta} & H_{i+1}(X) & \xrightarrow{\pi_*} & H_{i+1}(B) & \xrightarrow{e_F \cap} & H_{i-1}(B)
 \end{array}$$

For each integer  $0 \leq i \leq \dim X$  choose a basis  $\{b_1^i, \dots, b_{\beta_i}^i\}$  for  $H_i(X)$  such that for some integer  $m_i \leq \beta_i$   $\{b_{m_i+1}^i, \dots, b_{\beta_i}^i\}$  is a basis for the kernel of  $\pi_*: H_i(X) \rightarrow H_i(B)$ . The corresponding dual basis for  $H^i(X)$  will be denoted by  $\{\bar{b}_1^i, \dots, \bar{b}_{\beta_i}^i\}$ . Since we are using coefficients in a field, we make the identifications  $H_*(B \times S^1) \cong H_*(B) \otimes H_*(S^1)$  and  $H_*(X \times S^1) \cong H_*(X) \otimes H_*(S^1)$  via the natural isomorphism given by the homology exterior product. Let  $u \in H_1(S^1)$  be the generator determined by the standard orientation of  $S^1$ . Using Definition  $B_1$ ,

$$\chi_1(X; \mathbf{F})(\tau) = \sum_{k \geq 0} (-1)^{k+1} \sum_{j=1}^{\beta_k} \bar{b}_j^k \cap \Phi_*(b_j^k \otimes u).$$

Consider  $b_j^i \otimes u \in H_{i+1}(X \times S^1)$  where  $m_i + 1 \leq j \leq \beta_i$ . Since  $b_j^i$  lies in  $\ker \pi_*$ , the exactness of the Gysin sequence implies that  $b_j^i \otimes u = \theta'(c \otimes u)$  for some  $c \in H_i(B)$ . Consequently,

$$\Phi_*(b_j^i \otimes u) = \Phi_*(\theta'(c \otimes u)) = \theta(p_*(c \otimes u)) = 0$$

because  $p_*(c \otimes u) = 0$ . It follows that

$$(4.3) \quad \chi_1(X; \mathbf{F})(\tau) = \sum_{k \geq 0} (-1)^{k+1} \sum_{j=1}^{m_k} \bar{b}_j^k \cap \Phi_*(b_j^k \otimes u).$$

For each  $k$ , the set  $\{\pi_*(b_1^k), \dots, \pi_*(b_{m_k}^k)\}$  is a basis for the image of  $\pi_*: H_k(X) \rightarrow H_k(B)$ . Extend this set (in any manner) to basis for  $H_k(B)$  and let  $\{\pi_*(b_1^k), \dots, \pi_*(b_{m_k}^k)\}$  denote the corresponding portion of the dual basis for  $H^k(B)$ . Then  $\bar{b}_j^k = \pi^*(\pi_*(b_j^k))$ ,  $0 \leq j \leq m_k$ . Consider the commutative diagram:

$$\begin{array}{ccc}
 H^k(B \times S^1) & \xrightarrow{(\pi \times \text{id})^*} & H^k(X \times S^1) \\
 p^* \uparrow & & \Phi^* \uparrow \\
 H^k(B) & \xrightarrow{\pi^*} & H^k(X)
 \end{array}$$

Then, for  $0 \leq j \leq m_k$ ,

$$\begin{aligned}
 \bar{b}_j^k \cap \Phi_*(b_j^k \otimes u) &= \Phi_*(\Phi^*(\bar{b}_j^k) \cap (b_j^k \otimes u)) \\
 &= \Phi_*(\Phi^*(\pi^*(\overline{\pi_*(b_j^k)})) \cap (b_j^k \otimes u)) \\
 &= \Phi_*((\pi \times \text{id})^*(p^*(\overline{\pi_*(b_j^k)})) \cap (b_j^k \otimes u)) \\
 &\quad \text{using the above diagram} \\
 &= \Phi_*((\bar{b}_j^k \otimes 1) \cap (b_j^k \otimes u)) \\
 &= \Phi_*((\bar{b}_j^k \cap b_j^k) \otimes u) = \Phi_*({\{v\}} \otimes u) = {\{\tau\}}
 \end{aligned}$$

where  $\{v\}$  is the natural generator of  $H_0(X)$  determined by the inclusion of the basepoint  $v$  into  $X$ . From the proof of Lemma 4.1,  $\Phi_*({\{v\}} \otimes u) = {\{\tau\}}$ . Substituting the above computation into Formula 4.3 yields  $\chi_1(X; \mathbf{F})(\tau) = (\sum_{k \geq 0} (-1)^{k+1} m_k) {\{\tau\}}$ . If  $e_F \neq 0$  then Lemma 4.1 implies that  ${\{\tau\}} = 0$  and so  $\chi_1(X; \mathbf{F})(\tau) = 0$ . Thus the conclusion of the theorem is valid in this case. If  $e_F = 0$  then from the portion

$$H_k(X) \xrightarrow{\pi_*} H_k(B) \xrightarrow{e_F \cap} H_{k-2}(B)$$

of the Gysin homology sequence we deduce that  $\pi_*$  is onto and consequently  $m_k = \dim_{\mathbf{F}} H_k(B, \mathbf{F})$ . Thus  $\dim_{\mathbf{F}} H_*(B, \mathbf{F})$  is finite and  $\sum_{k \geq 0} (-1)^{k+1} m_k = -\chi(B; \mathbf{F})$ .  $\square$

Theorem 4.2 can be used to recalculate  $\chi_1(X; \mathbf{F})$  in Examples 3.8 and 3.9.

Next, we consider integer coefficients. Suppose that  $S^1 \rightarrow X \xrightarrow{\pi} B$  is a smooth orientable  $U(1)$ -bundle over a smooth, closed, oriented manifold  $B$ . Let  $\lambda$  be the one dimensional subbundle of the tangent bundle of  $X$  consisting of vectors which are tangent to the circle fibers and let  $v$  be a complementary bundle to  $\lambda$ . Then  $v \cong \pi^*(T_B)$  where  $T_B$  is the tangent bundle of  $B$ . Let  $[B] \in H_n(B; \mathbf{Z})$  be the fundamental class of  $B$  where  $n = \dim B$ . The Euler class,  $\text{Eul}(v) \in H^n(X; \mathbf{Z})$ , is given by

$$\text{Eul}(v) = \text{Eul}(\pi^*(T_B)) = \pi^*(\text{Eul}(T_B)) = \chi(B) \pi^*([B]^*)$$

where  $[B]^* \in H^n(B; \mathbf{Z})$  is the generator determined by the condition  $[B]^*([B]) = 1$ ; see [MS, Corollary 11.12]. The Gysin homology sequence for  $S^1 \rightarrow X \xrightarrow{\pi} B$  determines a fundamental class for  $X$ ;  $[X] \in H_{n+1}(X)$  is the image of  $[B]$  under the homomorphism  $\theta_n: H_n(B; \mathbf{Z}) \rightarrow H_{n+1}(X; \mathbf{Z})$ . For any closed oriented  $m$ -dimensional manifold  $M$ , let  $\text{PD}_M: H^i(M) \rightarrow H_{m-i}(M)$  be the Poincaré duality isomorphism explicitly given by  $\text{PD}_M(x) = (-1)^{i(m-i)} x \cap [M]$  where  $x \in H^i(M)$  and  $[M] \in H_m(M)$  is the

fundamental class  $((-1)^{i(m-i)})$  appears because of our use of Dold's sign conventions). An immediate consequence of Theorem 3.1 of [GN<sub>2</sub>] is the following computation of  $\chi_1(X)$  (with integer coefficients):

THEOREM 4.4.  $\chi_1(X)(\tau) = -\text{PD}_X(\text{Eul}(v))$ .  $\square$

THEOREM 4.5. *Under the above hypotheses,  $\chi_1(X)(\tau) = -\chi(B)\{\tau\}$ .*

*Proof.* There is a Poincaré duality isomorphism between the Gysin homology sequence and the Gysin cohomology sequence, a portion of which is shown below:

$$\begin{array}{ccccc} H_0(B; \mathbf{Z}) & \xrightarrow{\theta_0} & H_1(X; \mathbf{Z}) & \xrightarrow{\pi_*} & H_1(B; \mathbf{Z}) \\ \text{PD}_B \uparrow & & \text{PD}_X \uparrow & & \text{PD}_B \uparrow \\ H^n(B; \mathbf{Z}) & \xrightarrow{\pi^*} & H^n(X; \mathbf{Z}) & \rightarrow & H^{n-1}(B; \mathbf{Z}) \end{array}$$

Let  $v \in X$  be a basepoint, and let  $\{\pi(v)\} \in H_0(B; \mathbf{Z})$  be the generator determined by the inclusion of  $\pi(v)$  into  $B$ . From the above diagram,  $\text{PD}_X(\pi^*([B]^*)) = \theta_0(\{\pi(v)\})$ . Also, from the proof of Lemma 4.1,  $\theta_0(\{\pi(v)\}) = \{\tau\}$ . Thus  $\text{PD}_X(\text{Eul}(v)) = \chi(B)\{\tau\}$ . Regarding the free  $U(1)$ -action on  $X$  as a flow, we can now invoke Theorem 4.4 to conclude that  $\chi(B)\{\tau\} = -\chi_1(X)(\tau)$ .  $\square$

*Example 4.6.* Let  $\Sigma_g$  be a closed oriented surface of genus  $g > 1$  and let  $L_n$  be a complex line bundle over  $\Sigma_g$  with Chern number  $n$ . Let  $M_{n,g}$  be the total space of the  $U(1)$ -bundle associated to  $L_n$ . Then  $M_{n,g}$  is a closed oriented aspherical 3-manifold which fibers over  $\Sigma_g$ . The center of  $\pi_1(M_{n,g})$  is the infinite cyclic group generated by  $\tau$  (represented by a circle fiber); the image,  $\{\tau\}$ , of  $\tau$  in  $H_1(M_{n,g}) \cong \mathbf{Z}^{2g} \oplus \mathbf{Z}/n$  generates the  $\mathbf{Z}/n$  summand. By Theorem 4.5,  $\chi_1(M_{n,g}): \mathbf{Z} \rightarrow H_1(M_{n,g})$  is given by  $\chi_1(M_{n,g})(\tau) = (2g - 2)\{\tau\}$ .

Let  $T^n$ , where  $n > 1$ , be the  $n$ -torus (i.e. the  $n$ -fold product of copies of  $U(1)$ ). Let  $X$  be a closed oriented smooth manifold and let  $\rho: T^n \times X \rightarrow X$  be a smooth free action of  $T^n$ . This action defines a homomorphism  $\bar{\rho}: T^n \rightarrow \text{Diff}(X)$  where  $\text{Diff}(X)$  is the diffeomorphism group of  $X$ . Let  $\Gamma_\rho \subset \Gamma$  be the image of the composite:

$$\pi_1(T^n, 1) \xrightarrow{\bar{\rho}_\#} \pi_1(\text{Diff}(X), \text{id}) \rightarrow \pi_1(\mathcal{E}(X), \text{id}) = \Gamma.$$

PROPOSITION 4.7. *The restriction of  $\chi_1(X): \Gamma \rightarrow H_1(X)$  to  $\Gamma_\rho$  is the zero homomorphism.*

*Proof.* Since  $n > 1$ , if  $T \subset T^n$  is a circle subgroup then  $\chi(X/T) = 0$ . Applying Theorem 4.5 to the bundle  $T \rightarrow X \rightarrow X/T$  yields the conclusion.  $\square$

COROLLARY 4.8. *If  $n > 1$  then  $\chi_1(T^n): \mathbf{Z}^n \rightarrow \mathbf{Z}^n$  is zero.*  $\square$

## 5. A HIGHER ANALOG OF GOTTLIEB'S THEOREM

Let  $G$  be a group of type  $\mathcal{F}$ . Gottlieb's theorem (see Propositions 1.3 and 2.4) asserts that if  $\chi(G) \neq 0$  then  $Z(G)$ , the center of  $G$ , is trivial. We prove an analogous theorem for  $\chi_1(G; \mathbf{Q})$ : if  $\chi_1(G; \mathbf{Q}) \neq 0$  then the center of  $G$  is infinite cyclic provided  $G$  satisfies an extra hypothesis (explained below) related to the Bass Conjecture; see Proposition 5.2 and Theorem 5.4.

Throughout this section  $R$  will be a commutative ground ring. Let  $S$  be any associative  $R$ -algebra with unit. The Hochschild homology group  $HH_0(S)$  is the  $R$ -module  $S/[S, S]$  where  $[S, S]$  is the  $R$ -submodule of  $S$  generated by  $\{ab - ba \mid a, b \in S\}$ ; see §2. Recall that  $K_0(S)$  is the abelian group  $F/A$  where  $F$  is the free abelian group generated by the set of all isomorphism classes  $[M]$  of finitely generated projective right  $S$ -modules  $M \subset \bigoplus_{i=1}^{\infty} S$  and  $A$  is the subgroup of  $F$  generated by relations of the form  $[M_1 \oplus M_2] - [M_1] - [M_2]$ . Since a finitely generated projective module is the image of a finitely generated free module under an idempotent homomorphism, each element of  $K_0(S)$  can be represented by an idempotent matrix over  $S$ . The *Hattori-Stallings* trace  $T_0: K_0(S) \rightarrow HH_0(S)$  is defined as follows. Let  $A: M \rightarrow M$  be an idempotent endomorphism of a free, finitely generated right  $S$ -module  $M$  representing  $x \in K_0(S)$ . If  $[A]$  is the matrix of  $A$  with respect to a given basis for  $M$  then  $T_0(x)$  is defined to be  $T_0([A]) \in HH_0(S)$ .

Consider the groupring,  $RG$ , of a group  $G$  over  $R$ . Then  $HH_0(RG)$  is naturally isomorphic to the free  $R$ -module generated by  $G_1$ , the set of conjugacy classes of  $G$  (see §2 for an explanation in the case  $R = \mathbf{Z}$ ). Recall that for  $g \in G$  we write  $C(g) \in G_1$  for the conjugacy class of  $g$ ,  $HH_0(RG)_{C(g)}$  for the summand of  $HH_0(RG)$  corresponding to  $C(g)$  and  $x_{C(g)}$  for the  $C(g)$ -component of  $x \in HH_0(RG)$ . Also write  $HH_0(RG) = HH_0(RG)_{C(1)} \oplus HH_0(RG)'$  where  $1 \in G$  is the identity element of  $G$ , and  $HH_0(RG)'$  is the direct sum of the remaining summands. The augmentation homomorphism  $\varepsilon: RG \rightarrow R$  induces a homomorphism  $\varepsilon_*: HH_0(RG) \rightarrow HH_0(R) = R$ .