2. Borel's Theorem

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Although the cross ratio depends on the particular choice of the base point, the above homomorphism is independent of this choice (see [6, (4.10)]).

In the cusped case, we note that by [7], each cusped hyperbolic 3-manifold M has an ideal triangulation

$$M = \Delta_1 \cup \cdots \cup \Delta_n .$$

Let z_i be the cross ratio of any ordering of the vertices of Δ_i consistent with its orientation. The following relation was first discovered by W. Thurston (unpublished — see the remark in Zagier [21]; for a proof, see [14]):

$$\sum_i z_i \wedge (1 - z_i) = 0, \quad \text{in } \wedge^2_{\mathbf{Z}}(\mathbf{C}^*) .$$

This relation is independent of the particular choice of the ordering of the vertices of the Δ_i's.

From the definition of Bloch groups, it is thus clear that an ideal decomposition of a cusped hyperbolic 3-manifold determines an element $\beta(M) = \sum_i [z_i]$ in the Bloch group $\mathcal{B}(\mathbf{C})$. That such an element is independent of the choice of the ideal triangulation is proved in [14].

In fact, what we proved in [14] is stronger. It includes the following, which is what is needed for the purpose of this paper.

THEOREM 1.1. *Every hyperbolic 3-manifold M gives a well defined element $\beta(M)$ in the Bloch group $\mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$ which is the image of a well defined element $\beta_k(M)$ in $\mathcal{B}(k) \otimes \mathbf{Q}$ where k is the invariant trace field of M.* □

2. BOREL'S THEOREM

The Bloch group of an infinite field F is closely related to the third algebraic K-group of F (denoted by $K_3(F)$). There exists a natural map

$$\mathcal{B}(F) \to K_3(F)$$

due to Bloch [2]. In particular, if F is a number field, then there is the isomorphism

$$\mathcal{B}(F) \otimes \mathbf{Q} \cong K_3(F) \otimes \mathbf{Q} .$$

For the precise relationship between the Bloch group and K_3 of an infinite field, see the work of Suslin [19].

In [3], Borel generalized the classical Dirichlet Unit Theorem in classical number theory. The generalization started with the observation that for a number field F, the unit group of the ring of integers $\mathcal{O}_F$ is non-other than the first algebraic K-group of $\mathcal{O}_F$, $K_1(\mathcal{O}_F)$. So the generalization is along the line of higher K-groups. For the exact statement, we refer to [3]. For the purpose of this paper, we state Borel's Theorem for K_3 of a number field in terms of the Bloch group. The way we state Borel's theorem here therefore incorporates a series of important works by Bloch and Suslin.

Define the *Bloch-Wigner* function $D_2 : \mathbf{C} - \{0, 1\} \to \mathbf{R}$ by (cf. [2])

$$D_2(z) = \operatorname{Im} \ln_2(z) + \log|z| \arg(1 - z), \quad z \in \mathbf{C} - \{0, 1\}$$

where $\ln_2(z)$ is the classical dilogarithm function. The hyperbolic volume of an ideal tetrahedron Δ with cross ratio z is equal to $D_2(z)$. It follows that D_2 satisfies the five-term functional equation given by the 5-term relation, and therefore D_2 induces a map

$$D_2 : \mathscr{B}(\mathbf{C}) \to \mathbf{R},$$

by defining $D_2[z] = D_2(z)$.

Given a number field F, let r_1 and r_2 denote the number of real embeddings $F \subset \mathbf{R}$ and the number of pairs of conjugate complex embeddings $F \subset \mathbf{C}$ respectively. Let $\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \bar{\sigma}_{r_1+1}, \ldots, \sigma_{r_1+r_2}, \bar{\sigma}_{r_1+r_2}$ denote these embeddings. Given a number field F, then one has a map

$$\begin{aligned} c_2 : \quad & \mathscr{B}(F) \to \mathbf{R}^{r_2} \\ & \textstyle\sum_i (n_i[z_i]) \mapsto (\sum_i n_i D_2(\sigma_{r_1+1}(z_i)), \ldots, \sum_i n_i D_2(\sigma_{r_1+r_2}(z_i))) . \end{aligned}$$

BOREL'S THEOREM. *The kernel of* c_2 *is exactly the torsion subgroup of* $\mathscr{B}(F)$ *and the image of* c_2 *is a maximal lattice in* $\mathbf{R}^{r_2}$. *In particular, it follows that the rank of* $\mathscr{B}(F)$ *is* r_2. □

This theorem has some useful consequences. Denote $\mathscr{B}(F)_\mathbf{Q} = \mathscr{B}(F) \otimes \mathbf{Q}$ for short. An immediate consequence of the theorem is that an inclusion of number fields $F \hookrightarrow E$ induces an injection $\mathscr{B}(F)/Torsion \to \mathscr{B}(E)/Torsion$ and hence an injection $\mathscr{B}(F)_\mathbf{Q} \to \mathscr{B}(E)_\mathbf{Q}$. Note that if $F \subset E$ is a finite Galois extension of fields with Galois group H then H acts on $\mathscr{B}(E)$.

PROPOSITION 2.1. *For any subfield* F *of the number field* E *identify* $\mathscr{B}(F)_\mathbf{Q}$ *with its image in* $\mathscr{B}(E)_\mathbf{Q}$. *Then if* F_1 *and* F_2 *are two subfields we have*

$$\mathscr{B}(F_1 \cap F_2)_\mathbf{Q} = \mathscr{B}(F_1)_\mathbf{Q} \cap \mathscr{B}(F_2)_\mathbf{Q}$$

and if E/F *is a Galois extension with group* H *then*

$$\mathcal{B}(F)_{\mathbf{Q}} = (\mathcal{B}(E)_{\mathbf{Q}})^H$$

(the fixed subgroup of $\mathcal{B}(E)_{\mathbf{Q}}$ *under* H*).*

Proof. This result appears to be known to experts but we could find no published proof, so we provide one. It clearly suffices to prove the results for the Bloch groups tensored with $\mathbf{R}$ rather than with $\mathbf{Q}$. We first prove the Galois property in the case that E is Galois over $\mathbf{Q}$. Let $G = \mathrm{Gal}(E/\mathbf{Q})$. If $\tau : E \to \mathbf{C}$ is our given embedding and $\delta \in G$ the restriction of complex conjugation for this embedding, then all complex embeddings have the form $\tau \circ \gamma$ with $\gamma \in G$ and the conjugate embedding to $\tau \circ \gamma$ is $\tau \circ \delta\gamma$. Consider the map $\mathcal{B}(E) \otimes \mathbf{R} \to \mathbf{R}G$ given on generators by

$$[z] \mapsto \sum_{\gamma \in G} D_2(\tau(\gamma(z)))\gamma .$$

By Borel's theorem this is injective with image exactly

$$\left\{ \sum_{g \in G} r_\gamma \gamma \mid r_\gamma = -r_{\delta\gamma} \text{ for all } \gamma \in G \right\} .$$

The G-action on $\mathcal{B}(E)$ corresponds to the action of G from the right on $\mathbf{R}G$. Thus, if we identify $\mathcal{B}(E) \otimes \mathbf{R}$ with its image in $\mathbf{R}G$, then $\mathcal{B}(E)^H \otimes \mathbf{R}$ is identified with the set of elements $\sum r_\gamma \gamma \in \mathbf{R}G$ satisfying $r_\gamma = -r_{\delta\gamma}$ and $r_\gamma = r_{\gamma\theta}$ for all $\gamma \in G$ and $\theta \in H$. That is, the function r_γ is constant on right cosets of H and for the cosets γH and $\delta\gamma H$ it is zero if they coincide and otherwise takes opposite values on each. Thus the rank r of $\mathcal{B}(E)^H \otimes \mathbf{R}$ is the half the number of cosets γH for which $\gamma^{-1}\delta\gamma \notin H$. Now, with $F = E^H$, the embedding $\tau \circ \gamma | F$ depends only on the coset γH and is real or complex according as its conjugation map $\gamma^{-1}\delta\gamma$ does or does not lie in H, so the above rank r is just $r_2(F)$. Since the image of $\mathcal{B}(F) \otimes \mathbf{R}$ lies in $\mathcal{B}(E)^H \otimes \mathbf{R}$ and has this rank, it must be all of $\mathcal{B}(E)^H \otimes \mathbf{R}$.

The case when E is not Galois over $\mathbf{Q}$ now follows easily by embedding E in a larger field which is Galois over $\mathbf{Q}$. Similarly, for the intersection formula, by replacing E by a larger field as necessary we may assume E is Galois over F_1 and F_2 with groups H_1 and H_2 say. Then, identifying $\mathcal{B}(F_i)_{\mathbf{Q}}$ with its image in $\mathcal{B}(E)_{\mathbf{Q}}$ we have $\mathcal{B}(F_1)_{\mathbf{Q}} \cap \mathcal{B}(F_2)_{\mathbf{Q}} = \mathcal{B}(E)_{\mathbf{Q}}^{H_1} \cap \mathcal{B}(E)_{\mathbf{Q}}^{H_2} = \mathcal{B}(E)_{\mathbf{Q}}^{\langle H_1, H_2\rangle} = \mathcal{B}(E^{\langle H_1, H_2\rangle})_{\mathbf{Q}} = \mathcal{B}(F_1 \cap F_2)_{\mathbf{Q}}$. □