3. PROOF AND GENERALIZATION OF THEOREM B

We shall give a proof of Theorem B based directly on Borel's theorem. A different proof can be given using Proposition 2.1 and its consequence Theorem 3.1.

We denote the complex conjugation in $\mathbf{C}$ by δ. Let F be a fixed non-real subfield of $\mathbf{C}$ that is stable under complex conjugation, i.e., $\delta(F) = F$. Assume F is a finite extension field of $\mathbf{Q}$. Let r_2 be as in Borel's Theorem. Then we may list all the complex (non-real) embeddings of F into $\mathbf{C}$ as $\tau_1, \delta\tau_1, \ldots, \tau_{r_2}, \delta\tau_{r_2}$. Let r_2' be the number of conjugate pairs that commutes with δ, i.e., $\tau_i\delta = \delta\tau_i$. Renumbering if necessary, we may assume $\tau_1, \ldots, \tau_l$ are the ones that commute with δ. Note that by our assumption on F, r_2' is at least one. The rest of the τ's won't commute with δ, therefore τ_i and $\tau_i\delta$, $i > r_2'$ will be in different conjugate pairs (we use here that F is non-real). So renumbering if necessary, we may assume

$$\tau_1, \ldots, \tau_{r_2'}, \tau_{r_2'+1}, \tau_{r_2'+1}\delta, \ldots, \tau_m, \tau_m\delta$$

gives exactly one representative from each conjugate pair of embeddings of F into $\mathbf{C}$, where $m = r_2' + (r_2 - r_2')/2$.

The complex conjugation on F induces an involution on $\mathscr{B}(F)$. Let $\mathscr{B}_+(F)$ and $\mathscr{B}_-(F)$ be the $\pm$ eigenspace of $\mathscr{B}(F)_{\mathbf{Q}} = \mathscr{B}(F) \otimes \mathbf{Q}$. By Borel's Theorem, $\mathscr{B}(F)_{\mathbf{Q}}$ has a $\mathbf{Q}$-basis $\alpha_1, \ldots, \alpha_{r_2}$. Let

$$u_i = \alpha_i - \delta\alpha_i, \quad v_i = \alpha_i + \delta\alpha_i, \; 1 \leqslant i \leqslant r_2 .$$

Then u_i's and v_i's span $\mathscr{B}_-(F)$ and $\mathscr{B}_+(F)$ respectively. Together, they span $\mathscr{B}(F)_{\mathbf{Q}}$. Hence by Borel's Theorem, we know that the matrix

$$\begin{pmatrix}
D_2(\tau_1(u_1)) & \cdots & D_2(\tau_m(u_1)) & D_2(\tau_{r_2'+1}\delta(u_1)) & \cdots & D_2(\tau_m\delta(u_1)) \\
\vdots & & \vdots & \vdots & & \vdots \\
D_2(\tau_{r_2}(u_{r_2})) & \cdots & D_2(\tau_m(u_{r_2})) & D_2(\tau_{r_2'+1}\delta(u_{r_2})) & \cdots & D_2(\tau_m\delta(u_{r_2})) \\
D_2(\tau_1(v_1)) & \cdots & D_2(\tau_m(v_1)) & D_2(\tau_{r_2'+1}\delta(v_1)) & \cdots & D_2(\tau_m\delta(v_1)) \\
\vdots & & \vdots & \vdots & & \vdots \\
D_2(\tau_{r_2}(v_{r_2})) & \cdots & D_2(\tau_m(v_{r_2})) & D_2(\tau_{r_2'+1}\delta(v_{r_2})) & \cdots & D_2(\tau_m\delta(v_{r_2}))
\end{pmatrix}$$

has rank r_2. Note that because the first r_2' embeddings commute with δ, the entries of the last r_2 rows of the first r_2 columns are all 0's. Also, it follows from the equation $\delta(u_i) = -\delta(u_i)$ and $\delta(v_i) = \delta(v_i)$, this matrix has the following block form

$$\begin{pmatrix} A_{r_2\times r_2'} & B_{r_2\times(r_2-r_2')/2} & -B_{r_2\times(r_2-r_2')/2} \\ 0 & C_{r_2\times(r_2-r_2')/2} & C_{r_2\times(r_2-r_2')/2} \end{pmatrix}$$

So the matrix A has to have rank r_2'. For the last $(r_2 - r_2')$-columns to have rank $r_2 - r_2'$, the matrices B and C must both have maximal possible rank, that is, $(r_2 - r_2')/2$. Since by Borel's Theorem,

$$\operatorname{rank} C = \operatorname{rank} \mathscr{B}_+(F),$$

and $\operatorname{rank} \mathscr{B}_+(F) + \operatorname{rank} \mathscr{B}_-(F) = r_2$, Theorem B follows. $\square$

We can also describe the situation when $F \subset \mathbf{C}$ is a number field that is not stable under conjugation. If $E \subset \mathbf{C}$ is any number field containing F with $E = \bar{E}$ then $\mathscr{B}_+(E)$ and $\mathscr{B}_-(E)$ are defined, so we can form

$$\mathscr{B}_+(F) := \mathscr{B}_+(E) \cap \mathscr{B}(F)_{\mathbf{Q}} \quad \text{and} \quad \mathscr{B}_-(F) := \mathscr{B}_-(E) \cap \mathscr{B}(F)_{\mathbf{Q}}.$$

These subgroups are independent of the choice of E, but in general they will not sum to $\mathscr{B}(F)_{\mathbf{Q}}$.

Denote $F_{\mathbf{R}} = F \cap \mathbf{R}$ and let $F' = F \cap \bar{F}$. Clearly F' contains $F_{\mathbf{R}}$, and $F_{\mathbf{R}}$ must be the fixed field of conjugation on F'. Thus either $F' = F_{\mathbf{R}}$ or F' is an imaginary quadratic extension of $F_{\mathbf{R}}$. Now F' is a field to which Theorem B applies, so $\mathscr{B}(F')_{\mathbf{Q}} = \mathscr{B}_+(F') \oplus \mathscr{B}_-(F')$, with the ranks of the summands given by Theorem B.

THEOREM 3.1. *$\mathscr{B}_-(F) = \mathscr{B}_-(F')$ and $\mathscr{B}_+(F) = \mathscr{B}_+(F') = \mathscr{B}(F_{\mathbf{R}})_{\mathbf{Q}}$.*

COROLLARY 3.2. *$\mathscr{B}_+(F)$ is trivial if and only if $F_{\mathbf{R}}$ is totally real.*

$\mathscr{B}_-(F)$ is trivial if and only if $F' = F_{\mathbf{R}}$.

$\mathscr{B}_-(F) = \mathscr{B}(F)_{\mathbf{Q}}$ if and only if $F = F'$ and $F_{\mathbf{R}}$ is totally real; that is either F is totally real or the embedding $F \hookrightarrow \mathbf{C}$ is a CM-embedding.

Proof of Theorem 3.1. We work in a Galois superfield E of F and identify Bloch groups with their images in $\mathscr{B}(E)_{\mathbf{Q}}$. Let $G = \operatorname{Gal}(E/\mathbf{Q})$, so $H = \operatorname{Gal}(E/F) \subset G$ is the subgroup which fixes F. We fix an embedding $E \subset \mathbf{C}$ extending the given embedding of F and denote complex conjugation for this embedding by δ. Then the subgroup $H_{\mathbf{R}}$ generated by H and δ is $\operatorname{Gal}(E/F_{\mathbf{R}})$, so it follows from Proposition 2.1 that $\mathscr{B}_+(F) = \mathscr{B}(E^H)_{\mathbf{Q}} \cap \mathscr{B}(E)_{\mathbf{Q}}^{\delta} = (\mathscr{B}(E)_{\mathbf{Q}}^{H})^{\delta} = \mathscr{B}(E)_{\mathbf{Q}}^{H_{\mathbf{R}}} = \mathscr{B}(F_{\mathbf{R}})_{\mathbf{Q}}$. Moreover, $\mathscr{B}_-(F)$ is fixed by both H and $\delta H \delta$ and hence by the group H' that they generate. But H' is the Galois group in E of $F \cap \delta(F) = F \cap \bar{F} = F'$. Thus $\mathscr{B}_-(F)$ is in

$\mathcal{B}(E)^{H'}_{\mathbf{Q}} = \mathcal{B}(F')_{\mathbf{Q}}$. We thus obtain inclusions $\mathcal{B}_{\pm}(F) \subset \mathcal{B}_{\pm}(F')$, and the reverse inclusions are trivial. □

Proof of Corollary 3.2. The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that $\mathcal{B}_{-}(F) \subset \mathcal{B}(F')_{\mathbf{Q}}$ and if $F' \neq F$ then F has strictly more complex embeddings than F' so $\mathcal{B}(F')_{\mathbf{Q}} \neq \mathcal{B}(F)_{\mathbf{Q}}$. Thus, to have $\mathcal{B}_{-}(F) = \mathcal{B}(F)_{\mathbf{Q}}$ we must have $F = F'$. The claim then follows directly from Theorem B. □

REMARK. We have pointed out at the beginning of sect. 2 that $\mathcal{B}(F)$ could have been replaced by $K_3(F)$ in all our discussions. The analog of Borel's theorem holds for $K_i(F)$ for all $i \equiv 3 \pmod 4$, so the results described above are also valid for these K-groups. When $1 < i \equiv 1 \pmod 4$ Borel's theorem gives a map $K_i(F) \to \mathbf{R}^{r_1+r_2}$ whose kernel is torsion and whose image is a lattice. The only change is that one obtains $r_1 + \frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$ as the dimensions of the $+$ and $-$ eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if $E \subset \mathbf{C}$ is Galois over $\mathbf{Q}$ with group G and δ is its conjugation then $K_i(E) \otimes \mathbf{R}$ is G-equivariantly isomorphic to $\{\sum r_\gamma \gamma \in \mathbf{R}G \mid r_\gamma = (-1)^{(i-1)/2} r_{\delta\gamma}\}$ for $i > 1$ and odd.

4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that $D_2(z)$ represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer $N \geqslant 3$, the real numbers $D_2(e^{2\pi\sqrt{-1}j/N})$, with j relatively prime to N and $0 < j < N/2$, are linearly independent over the rationals.*

A field homomorphism $\tau: F \to K$ clearly induces a homomorphism on the Bloch groups $\mathcal{B}(F) \to \mathcal{B}(K)$ which, by abuse of notation, will again be denoted by τ.

Given a cyclotomic field $F = \mathbf{Q}(e^{2\pi\sqrt{-1}/N})$, the elements $[e^{2\pi ij/N}]$, with j relatively prime to N and $0 < j < N/2$, form a basis of the Bloch group $\mathcal{B}(F) \otimes \mathbf{Q}$ (see Bloch [2]). Hence Milnor's conjecture can be reformulated that $D_2: \mathcal{B}(F) \to \mathbf{R}$ given on generators by $[z] \mapsto D_2(z)$ is injective for a cyclotomic field F.