5. Chern-Simons invariants and the regulator map

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As we have seen from the above discussion, hyperbolic 3-manifolds and their volumes all have interesting K-theoretic interpretation. Another important invariant in the theory of hyperbolic 3-manifolds is the Chern-Simons invariant of [4] and [9]. What is its relation with K-theory? As discussed in sect. 1, any hyperbolic 3-manifold M represents a class $\beta(M)$ in $\mathcal{B}(\mathbf{C})$. For M compact, the next theorem follows from Theorem 1.11 of Dupont [5]. A proof for cusped M can be found in [14].

Theorem 5.1. *The Chern-Simons invariant of M mod $\mathbf{Q}(2)$ is equal to the real part of the image of $\beta(M)$ under the regulator map*

$$\rho : \mathcal{B}(\mathbf{C}) \to \mathbf{C}/\mathbf{Q}(2)\,. \qquad \square$$

Now given a hyperbolic 3-manifold M with invariant trace field k and the associated embedding $\sigma : k \hookrightarrow \mathbf{C}$, the class $\beta(M) \in \mathcal{B}(\mathbf{C})$ associated to M is in the image of $\mathcal{B}(k)_{\mathbf{Q}} \xrightarrow{\sigma} \mathcal{B}(\mathbf{C})$ (Theorem 1.1). If σ is a CM-embedding, then it follows from Corollary 3.2 that the real part of ρ is trivial on $\mathcal{B}(F) \otimes \mathbf{Q}$. Theorem A in the introduction now follows immediately from Theorem 5.1. Similarly, as in the previous section, Corollary 3.2 implies that the irrationality conjecture for Chern Simons invariant of the Introduction would be implied by the Ramakrishnan Conjecture.

We have seen that the imaginary part of ρ is essentially the volume map while the real part of ρ can be called the Chern-Simons map. Reinterpreting the discussion of the previous section, this means that if $k = \bar{k} \subset \mathbf{C}$, then the volume map on $\mathcal{B}(k)$ factors through $\mathcal{B}_-(k)$ and Chern-Simons map on $\mathcal{B}(K)$ factors over $\mathcal{B}_+(k)$. By Theorem B we thus get bounds of $\frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$ on the number of rationally independent volumes resp. Chern-Simons invariants for manifolds having invariant trace field contained in our given k. Ramakrishnan's Conjecture says the image of ρ has $\mathbf{Q}$-rank r_2. This is equivalent to the conjecture that the $\mathbf{Q}$-ranks of the images of vol: $\mathcal{B}(k) \to \mathbf{R}$ and CS: $\mathcal{B}(k) \to \mathbf{R}/\mathbf{Q}$ are exactly $\frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$.