1. Topological classification of certain 6-manifolds

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It is related to the well-known inequality $c_1^2 \leqslant 3c_2$ and has been solved to a considerable extent.

Though in the case of 6-folds the corresponding question about the realisability of cubic forms is definitely weaker than the question which 6-folds carry a complex or algebraic structure, it still remains of much interest. In the second half of this paper we say something about algebra and arithmetic of cubic forms and consider the apparently largely untouched question of the realisability of *complex* forms by complex manifolds. Apart from a considerable number of examples some conditions for Kähler manifolds are given. And to show how few 6-folds of the type in question actually carry Kähler structures, we add a theorem about Kähler structures on the set of 6-folds with $b_2 = 1$, $b_3 \leqslant$ constant and $w_2 \neq 0$.

The first part of this paper surveys the results of Wall and Jupp referred to before, and deals with the homotopy classification. By putting together (for the first time?) all this in a rather systematic way we hope to contribute to the knowledge of complex 3-folds from a topological point of view.

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1. Topological classification of certain 6-manifolds

The topological classification of 1-connected, closed, oriented, 6-dimensional manifolds has been developped in a sequence of papers by C.T.C. Wall [W], P. Jupp [J], and A. Žubr [Z1], [Z2], [Z3]. Roughly speaking, their main result is that the topological classification of these 6-manifolds is equivalent to the arithmetic classification of certain systems of invariants naturally associated with them.

The aim of this section is to review these results and to reformulate the arithmetic classification problem in a way which makes it accessible to further investigation.

1.1 Homeomorphism types and C^∞-structures

Let X be a closed, oriented, 6-dimensional topological manifold; we assume that X is 1-connected with torsion-free homology. The *basic invariants* of X are [J]:

i) $H^2(X, \mathbf{Z})$, a finitely generated free abelian group;

ii) $b_3(X) = rk_{\mathbf{Z}} H^3(X, \mathbf{Z})$, a natural number which is even since $H^3(X, \mathbf{Z})$ admits a non-degenerate symplectic form;

iii) $F_X: H^2(X, \mathbf{Z}) \otimes H^2(X, \mathbf{Z}) \otimes H^2(X, \mathbf{Z}) \to \mathbf{Z}$, a symmetric trilinear form given by the cup-product evaluated on the orientation class;

iv) $p_1(X) \in H^4(X, \mathbf{Z})$, the first Pontrjagin class which is always integral because the inclusion of BO in $BTOP$ induces an isomorphism $H^4(BTOP, \mathbf{Z}) \to H^4(BO, \mathbf{Z})$ [J];

v) $w_2(X) \in H^2(X, \mathbf{Z}_{/2})$, the second Stiefel-Whitney class; $w_2(X)$ is determined by the Steenrod square $Sq^2: H^4(X, \mathbf{Z}_{/2}) \to H^6(X, \mathbf{Z}_{/2})$, $Sq^2(\xi) = w_2(X) \cdot \xi \ \forall \xi \in H^4(X, \mathbf{Z}_{/2})$ [W];

vi) $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$, the triangulation class which is the obstruction to lifting the stable tangent bundle of Y to a PL bundle [J].

These invariants satisfy one *fundamental relation*

$$(*) \qquad W^3 \equiv (p_1(X) + 24T) \cdot W (\text{mod } 48)$$

for all integral classes $W \in H^2(X, \mathbf{Z})$, $T \in H^4(X, \mathbf{Z})$ with $\bar{W} \equiv w_2(X) (\text{mod } 2)$, $\bar{T} \equiv \tau(x) (\text{mod } 2)$.

For smooth manifolds (*) is simply the $\hat{A}$-integrality theorem of A. Borel and F. Hirzebruch [B/H], whereas for topological manifolds additional surgery arguments are necessary [J].

In the sequel we shall use Poincaré duality to identify $H^4(X, \mathbf{Z})$ with $\text{Hom}_{\mathbf{Z}}(H^2(X, \mathbf{Z}), \mathbf{Z})$, so that $p_1(X)$ can be considered as a linear form on $H^2(X, \mathbf{Z})$, and we will write $x \cdot y \cdot z$ instead of $F_X(x \otimes y \otimes z)$ for elements $x, y, z \in H^2(X, \mathbf{Z})$.

DEFINITION 1. *A system of invariants is a 6-tuple* (r, H, w, τ, F, p) *consisting of a non-negative integer* r, *a finitely generated free abelian group* H, *elements* $w \in H/_{2H}$ *and* $\tau \in H^\vee/_{2H^\vee}$, *a symmetric trilinear form* $F \in S^3 H^\vee$, *and a linear form* $p \in H^\vee$. *The system* (H, r, w, τ, F, p) *is admissible iff for every* $W \in H$ *and* $T \in H^\vee$ *with* $\bar{W} \equiv w (\text{mod } 2)$ *and* $\bar{T} \equiv \tau (\text{mod } 2)$ *the following congruence holds:*

$$(*) \qquad W^3 \equiv (p + 24T)(W) (\text{mod } 48) .$$

Two systems of invariants (H, r, w, τ, F, p) *and* $(H', r', w', \tau', F', p')$ *are equivalent iff* $r = r'$, *and there exists an isomorphism* $\alpha: H \to H'$ *such that:*

$$\alpha(w) = w', \quad \alpha^*(\tau') = \tau, \quad \alpha^*(F') = F, \quad \alpha^*(p') = p .$$

The main classification result can now be formulated in the following way:

THEOREM 1 (Jupp). *The assignment*

$$X \mapsto \left(\tfrac{b_3(X)}{2}, H^2(X,\mathbf{Z}), w_2(X), \tau(X), F_X, p_1(X)\right)$$

induces a 1-1 correspondence between oriented homeomorphism classes of 1-connected, closed, oriented, 6-dimensional topological manifolds with torsion-free homology, and equivalence classes of admissible systems of invariants.

Furthermore, a topological manifold X *as above admits a* C^∞*-structure if and only if the triangulation class* $\tau(X)$ *vanishes; the* C^∞*-structure is then unique.*

REMARK 1. The classification theorem is due to C.T.C. Wall in the special case of differentiable spin-manifolds [W]; the final form above was obtained by P. Jupp [J].

A. Žubr generalized Wall's result in another direction; he proved a classification theorem for 1-connected, smooth spin-manifolds with not necessarily torsion-free homology [Z1]; in two further papers [Z2], [Z3] he also obtains P. Jupp's classification, and he asserts in addition, that algebraic isomorphisms of systems of invariants can always be realized by orientation preserving homeomorphisms (diffeomorphisms in the smooth case).

Note that the first invariant $\frac{b_3(X)}{2}$ of the system is completely independent of the remaining invariants, so that the following splitting theorem holds:

COROLLARY 1. *Every 1-connected, closed, oriented, 6-dimensional, topological (differentiable) manifold* X *with torsion-free homology admits a topological (differentiable) splitting* $X = X_0 \# \frac{b_3(X)}{2}(S^3 \times S^3)$ *as a connected sum of a core* X_0 *with* $b_3(X_0) = 0$, *and* $\frac{b_3(X)}{2}$ *copies of* $S^3 \times S^3$. *The oriented homeomorphism (diffeomorphism) type of* X_0 *is unique.*

EXAMPLE 1. The 1-connected, closed, oriented 6-manifolds X with $H_2(X,\mathbf{Z}) = 0$ are S^6 and the connected sums $\#_r S^3 \times S^3$ of $r \geqslant 1$ copies of $S^3 \times S^3$ [*Sm*].

1.2 HOMOTOPY TYPES

In order to describe the homotopy classification of the 6-manifolds above, we need some more preparations.

Let (H, F) be a pair consisting of a finitely generated free abelian group H, and a symmetric trilinear form F; consider the following subgroup of $H^\vee/_{48H^\vee}$:

$$U_F := \{l \in H^\vee/_{48H^\vee} \mid \exists u \in H \text{ with } l(x) \equiv 24u^2 \cdot x (\text{mod } 48)\ \forall x \in H\}\ .$$

If (H', F') is another such pair, and $\alpha : H \to H'$ an isomorphism with $\alpha^*(F') = F$, then there is an induced isomorphism

$$\alpha^* : H'^\vee/_{48H'^\vee}/_{U'_F} \to H^\vee/_{48H^\vee}/_{U_F}$$

of the quotients. Denote the class of a linear form $l \in H^\vee$ in the quotient $H^\vee/_{48H^\vee}/_{U_F}$ by $[l]$.

DEFINITION 2. *Two systems of invariants* (r, H, w, τ, F, p) *and* $(r', H', w', \tau', F', p')$ *are weakly equivalent iff* $r = r'$, *and there exists an isomorphism* $\alpha : H \to H'$ *such that:*

$$\alpha(w) = w', \alpha^*(F') = F, \quad \textit{and} \quad \alpha^*[p' + 24T'] = [p + 24T]$$
$$\textit{for all} \quad T \in H^\vee, T' \in H'^\vee \quad \textit{with} \quad \bar{T} \equiv \tau (\text{mod } 2), \bar{T}' \equiv \tau' (\text{mod } 2)\ .$$

With this definition we can phrase the homotopy classification in the following way:

THEOREM 2 (Žubr). *The assignment*

$$X \to \left(\tfrac{b_3(X)}{2}, H^2(X, \mathbf{Z}), w_2(X), \tau(X), F_X, p_1(X)\right)$$

induces a 1-1 correspondence between oriented homotopy classes of 1-connected, closed, oriented, 6-dimensional topological manifolds with torsion-free homology and weak equivalence classes of admissible systems of invariants.

REMARK 2. Žubr's theorem corrects and generalizes the homotopy classification in the papers by Wall [W] and Jupp [J]; he also treats manifolds with not necessarily torsion-free homology, and states without proof that algebraic isomorphisms of weak equivalence classes of systems of invariants are always realizable by orientation preserving homotopy equivalences [Z3].

EXAMPLE 2. Manifolds with $b_2(X) = 1$.

Let X be a 1-connected, closed, oriented, 6-dimensional manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$. Splitting off possible copies of $S^3 \times S^3$ we may assume $b_3(X) = 0$. Choosing a $\mathbf{Z}$-basis of $H^2(X, \mathbf{Z})$ we see that systems of invariants can be identified with 4-tuples $(\bar{W}, \bar{T}, d, p) \in \mathbf{Z}_{/2} \times \mathbf{Z}_{/2} \times \mathbf{Z} \times \mathbf{Z}$

where the 'degree' d corresponds to the cubic form. Such a 4-tuple is admissible iff $d(2x + \bar{W})^3 \equiv (p + 24\bar{T}) \cdot (2x + \bar{W}) \pmod{48}$ holds for every integer x. This is equivalent to $p \equiv 4d \pmod{24}$ if $\bar{W} = 0$, and to $p \equiv d + 24\bar{T} \pmod{48}$ with $d \equiv 0 \pmod 2$ if $\bar{W} \neq 0$.

Two admissible 4-tuples $(\bar{W}, \bar{T}, d, p)$ and $(\bar{W}', \bar{T}', d', p')$ are equivalent iff $\bar{W}' = \bar{W}, \bar{T}' = \bar{T}$ and $(d', p') = \pm (d, p)$. Taking the degree d non-negative, we find:

PROPOSITION 1. *There is a 1-1 correspondence between oriented homeomorphism types of cores* X_0 *with* $b_2(X_0) = 1$, *and 4-tuples* $(\bar{W}, \bar{T}, d, p)$, *normalized so that* $d \geqslant 0$, *and* $p \geqslant 0$ *if* $d = 0$, *which satisfy* $p \equiv 4d \pmod{24}$ *if* $\bar{W} = 0$, *and* $d \equiv 0 \pmod 2$, $p \equiv d + 24\bar{T} \pmod{48}$ *if* $\bar{W} \neq 0$.

In order to classify the associated homotopy types we first have to determine the subgroup U_F associated to a given cubic form F. By definition we find $U_F = 0$ if $d \equiv 0 \pmod 2$, $U_F = \mathbf{Z}_{/2}$ if $d \equiv 1 \pmod 2$. Two normalized 4-tuples $(\bar{W}, \bar{T}, d, p)$ and $(\bar{W}', \bar{T}', d', p')$ are weakly equivalent iff $d' = d, \bar{W}' = \bar{W}$, and $p + 24\bar{T} \equiv p' + 24\bar{T}' \pmod{48}$ if $d \equiv 0 \pmod 2$, $p \equiv p' \pmod{24}$ if $d \equiv 1 \pmod 2$.

Putting everything together, we find a single oriented homotopy type for every odd degree $d \geqslant 0$, which is necessarily spin, and 3 oriented homotopy types for every even degree $d \geqslant 0$; one of these 3 types has $\bar{W} \neq 0$, the other two are spin, and they are distinguished by $p + 24\bar{T} \pmod{48}$ i.e. $p \equiv 4d \pmod{48}$, or $p \equiv 4d + 24 \pmod{48}$.

2. REALIZATION OF CUBIC FORMS

In the previous section the (homotopy) topological classification of 1-connected, closed, oriented, 6-dimensional manifolds with torsion-free homology has been transformed into an arithmetical moduli problem: to describe the sets of (weak) equivalence classes of admissible systems of invariants. In this section we begin to investigate the latter problem; we give a simple criterion for the realizability of cubic forms by smooth manifolds, and we describe, at least in principle, the classification of homotopy types of manifolds with a given cohomology ring.