6. Mapping Tori

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(a) The restriction of $\chi_1(G; \mathbf{Q})$ to $Z(G) \cap [G, G]$ is zero.

(b) If $\chi_1(G; \mathbf{Q}) \neq 0$ then $\dim_{\mathbf{Q}} A_{\mathbf{Q}}(Z(G)) = 1$.

The desired conclusion follows easily from (a), (b) and Theorem 4.2.

Theorem 5.7 raises the question: For what groups G of type $\mathscr{F}$ is $\chi_1(G, \mathbf{Q}) \neq 0$? We give a necessary condition. Recall that a group H has type $\mathscr{F}\mathscr{D}$ if there is a finitely dominated $K(H, 1)$ (i.e. $K(H, 1)$ is a homotopy retract of a finite complex).

PROPOSITION 5.8. *If* $\chi_1(G, \mathbf{Q}) \neq 0$ *then* G *is isomorphic to a semidirect product* $\langle H, t \mid tht^{-1} = \theta(h)$ *for all* $h \in H \rangle$ *where* H *has type* $\mathscr{F}\mathscr{D}$.

Proof. Let $\tau \in Z(G)$ be such that $\chi_1(G, \mathbf{Q})(\tau) \neq 0$. By Theorem 4.2, it follows that $\{\tau\} \in H_1(G) \equiv G_{ab}$ is of infinite order. Thus there is an epimorphism $p: G \to \mathbf{Z}$ with $p(\tau) = n$ for some $n > 0$. Let $H = \ker(p)$. Since $\tau \in Z(G)$, $p^{-1}(n\mathbf{Z}) \cong H \times \mathbf{Z}$ and has finite index in G. Thus $H \times \mathbf{Z}$ has type $\mathscr{F}$ and so H has type $\mathscr{F}\mathscr{D}$. $\square$

Thus it is worthwhile to compute $\chi_1(G, \mathbf{Q})$ in terms of such a semidirect product structure. The geometric problem underlying this is the study of $\chi_1(X)$ where X is a mapping torus. We study this next, returning to the group theoretic case in §7.

6. MAPPING TORI

In this section, we consider $\chi_1(X)$ and $\tilde{\chi}_1(X)$ when X is the mapping torus of a map $f: Z \to Z$. The main results are Theorems 6.3, 6.13, 6.14, 6.16 and Corollary 6.18. Applications to the aspherical case will be given in §7.

Suppose Z is a path connected space and has a basepoint $v \in Z$. Given a continuous map $f: Z \to Z$, its *mapping torus*, denoted by $T(Z, f)$, is the space obtained from $Z \times [0, 1]$ by identifying $(z, 1)$ with $(f(z), 0)$ for each $z \in Z$. The image of $(z, u) \in Z \times [0, 1]$ in $T(Z, f)$ will be denoted by $[z, u]$. Choose a basepath σ from v to $f(v)$ and let $\theta: H \to H$ be the self homomorphism of $H \equiv \pi_1(Z, v)$ determined by f and σ.

Let $X = T(Z, f)$. Choose $w = [v, 0]$ as a basepoint for X and let $G = \pi_1(X, w)$. There is a canonical map of X to the standard circle S^1 (realized as complex numbers of unit modulus) given by: $p_f: X \to S^1$, $p_f([z, s]) = e^{2\pi i s}$. Let $i: Z \hookrightarrow X$ be the inclusion $z \mapsto [z, 0]$.

Recall that $\Gamma = \pi_1(\mathscr{E}(X), \mathrm{id})$. Let $\Gamma_{S^1} = \pi_1(\mathscr{E}(S^1), \mathrm{id})$. Let $\bar{\gamma}: I \to X$ be the path $\bar{\gamma}(u) = [v, u]$ and let $\gamma_0: I \to X$ be the path $\gamma_0 = \bar{\gamma}(i \circ \sigma)^{-1}$. Define a continuous map $P: X^X \to (S^1)^{S^1}$ by $P(g)(e^{2\pi i u}) = p_f(g(\gamma_0(u)))$. Then P induces a homomorphism $P_*: \Gamma \to \Gamma_{S^1}$. We define an identification $\Gamma_{S^1} \overset{\cong}{\to} \mathbf{Z}$ by sending the generator $[s \mapsto (e^{2\pi i u} \mapsto e^{2\pi i(u+s)})] \in \Gamma_{S^1}$ to $1 \in \mathbf{Z}$. The *rotation degree* of $\gamma \in \Gamma$ is the integer $P_*(\gamma)$.

We now describe some useful homotopies of X.

For a non-negative integer k, the *k-th tumble* is the homotopy which "rolls the mapping torus through an angle of $2\pi k$"; explicitly, this homotopy, denoted by $R_k: X \times [0, 1] \to X$, is given by the formula $R_k([z, u], s) = [f^{[ks+u]}(z), (ks + u) \bmod 1]$ where $[ks + u]$ is the integer part of $ks + u$.

Whenever a map $g: Z \to Z$ commutes with f (i.e. $fg = gf$), there is an induced "level" map $\hat{g}: X \to X$ given by $\hat{g}([z, u]) = [g(z), u]$; for example, the k-th tumble, R_k, is a homotopy from id_X to $\hat{f}^k$. We need a more general procedure (see Proposition 6.2 below) for extending homotopies of Z to homotopies of X.

A homotopy $N: Z \times I \to Z$ *eventually commutes* with f if there exists an integer $m \geqslant 0$ and a homotopy $J: Z \times I \times I \to Z$ with $J(z, u, 0) = f^m \circ N(f(z), u)$, $J(z, u, 1) = f^{m+1} \circ N(z, u)$, $J(z, 0, s) = f^m \circ N(f(z), 0)$, $J(z, 1, s) = f^m \circ N(f(z), 1)$. Thus J makes the following diagram commute up to homotopy rel $Z \times \{0, 1\} \times I$:

$$
\begin{array}{ccccc}
Z \times I & \overset{f \times \mathrm{id}}{\to} & Z \times I & \overset{N}{\to} & Z \\
\downarrow N & & & & \downarrow f^m \\
Z & \overset{f^{m+1}}{\to} & Z & = & Z
\end{array}
\qquad (6.1)
$$

This implies $f^m \circ N_i \circ f = f^{m+1} \circ N_i$ for $i = 0, 1$; in our applications, N_0 and N_1 will be iterates of f.

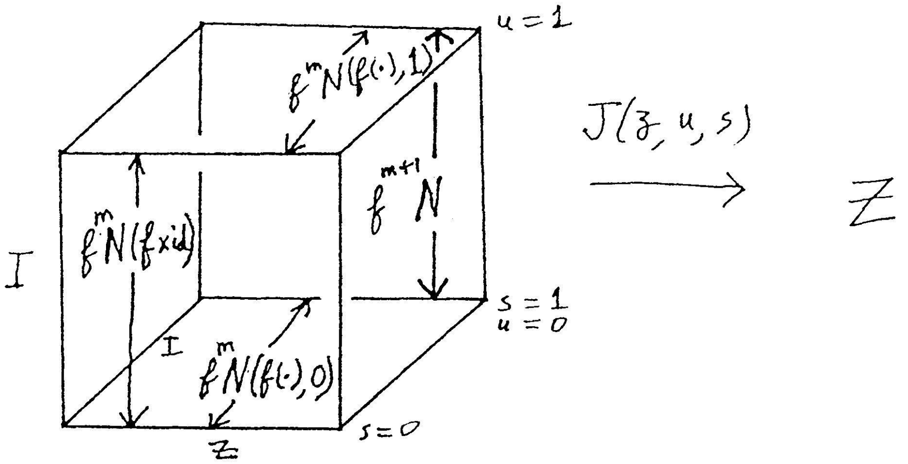

FIGURE 1

Define $L'_{(N,J,m)}: X \times I \to X$ (abbreviated to L') by the formula:

$$L'([z,u],s) = \begin{cases} [f^m \circ N(z,s), 2u] & \text{if } 0 \leqslant u \leqslant \frac{1}{2} \\ [J(z,s,2-2u),0] & \text{if } \frac{1}{2} \leqslant u \leqslant 1 \,. \end{cases}$$

and define $K: X \times I \to X$ by:

$$K([z,u],s) = \begin{cases} [z, u(1+s)] & \text{if } 0 \leqslant u \leqslant \frac{1}{2} \\ [z, s(1-u)+u] & \text{if } \frac{1}{2} \leqslant u \leqslant 1 \,. \end{cases}$$

(K is a "linear" homotopy from id_X to a map which sends the points $[z,u]$, $\frac{1}{2} \leqslant u \leqslant 1$, to $[z,1] \equiv [f(z),0]$.)

Observe that $L'(\,\cdot\,,0) = K(\,\cdot\,,1) \circ (\widehat{f^m \circ N_0})$ and $L'(\,\cdot\,,1) = K(\,\cdot\,,1) \circ (\widehat{f^m \circ N_1})$. Thus, for N, J and m as above, we have:

PROPOSITION 6.2. *The concatenation*

$$L_{(N,J,m)} \equiv K \circ (\widehat{f^m \circ N_0} \times \mathrm{id}) \bigstar L'_{(N,J,m)} \bigstar (K \circ (\widehat{f^m \circ N_1} \times \mathrm{id}))^{-1}$$

is a homotopy from $\widehat{f^m \circ N_0}$ *to* $\widehat{f^m \circ N_1}$. □

(For a homotopy Q, Q^{-1} means the homotopy $Q^{-1}(x,s) = Q(x,1-s)$.)

Next, we will build special elements of Γ. The map $f: Z \to Z$ is a *periodic homotopy idempotent* if there exists $r \geqslant 0$ and $q > 0$ such that f^r is homotopic to f^{r+q}; it is not assumed that this can be achieved by a basepoint preserving homotopy. If for some $r \geqslant 0$ and $q > 0$ there is a homotopy $N: f^r \simeq f^{r+q}$ for which there exist J and $m \geqslant 0$ making Diagram 6.1, commute up to homotopy rel $Z \times \{0,1\} \times I$, then we say that f is *eventually coherent*. In this case, Proposition 6.2 gives a homotopy $L_{(N,J,m)}: \hat{f}^{r+m} \simeq \hat{f}^{r+q+m}$. The concatenation $S \equiv S_{(r,N,J,m)} \equiv R_{r+q+m} \bigstar L^{-1}_{(N,J,m)} \bigstar R^{-1}_{r+m}$ is a homotopy from id_X to id_X whose rotation degree is q. Given f, the least $q > 0$ for which there exist r, N, J and m as above (assuming that they exist at all) is the *period* of f. Then r and m may be chosen as large as desired.

These conditions on a map f which give rise to an element $[S] \in \Gamma$ having positive rotation degree, are not arbitrary. Rather, they are the general case:

THEOREM 6.3. *Let* $f: Z \to Z$ *be a map for which the rotation degree homomorphism* $P_*: \Gamma \to \mathbf{Z}$ *is non-zero. Let* q *be the least positive element of* $P_*(\Gamma)$. *Then* f *is an eventually coherent periodic homotopy idempotent of period* q.

Before proving this, we set up notation for points of the infinite mapping telescope of f, i.e. the infinite cyclic cover of X whose fundamental group is H. This space, denoted by $\bar{X}$, is the quotient of the disjoint union $\coprod_{n \in \mathbf{Z}} Z \times \{n\} \times [0, 1]$ obtained by identifying $(z, n, 1)$ with $(f(z), n+1, 0)$ for all n. The image of (z, n, u) in $\bar{X}$ will be denoted by $[z, n, u]$. The covering projection $\bar{X} \to X$ is given by $[z, n, u] \mapsto [z, (n+u) \bmod 1]$. The space $\bar{X}$ is a "two-ended union" of mapping cylinders: we write $M(f)_n$ for the subset of points $[z, n, u]$ such that $0 \leqslant u \leqslant 1$, and Z_n for the subset of points $[z, n, 0]$.

Proof of 6.3. Let $F^\gamma : \mathrm{id}_X \simeq \mathrm{id}_X$ represent $\gamma \in \Gamma$ of rotation degree q, and let $\bar{F}^\gamma : \bar{X} \times I \to \bar{X}$ be the basepoint preserving lift of F^γ. The map $\bar{F}^\gamma$ is a homotopy between $\mathrm{id}_{\bar{X}}$ and $\mathscr{T}^q$, where $\mathscr{T}([z, n, u]) = [z, n+1, u]$ is "translation by 1". Let $i_n : Z \to \bar{X}$ be the "inclusion" of Z as Z_n, i.e. $i_n(z) = [z, n, 0]$. The composition $Z \times I \xrightarrow{i_0 \times \mathrm{id}} \bar{X} \times I \xrightarrow{\bar{F}^\gamma} \bar{X}$ gives a homotopy between i_0 and i_q. The formula $(z, s) \mapsto [f^{[sq]}(z), [sq], sq - [sq])$ gives a homotopy between i_0 and $i_q \circ f^q$. Combining the two, we get a homotopy $\Phi : i_q \simeq i_q \circ f^q$. The track of $\Phi : Z \times I \to \bar{X}$ lies in $\cup_{n=r'}^{r+q-1} M(f)_n$ for suitable integers $r' \leqslant 0 < q \leqslant r+q$. Form a homotopy $\Psi : Z \times I \to \bar{X}$, $\Psi : i_{r+q} \circ f^r \simeq i_{r+q} \circ f^{r+q}$, whose entire image lies in Z_{r+q}, by "pushing" the track of Φ along the mapping telescope into Z_{r+q}; explicitly, if $\Phi(z, s) = [z', n', u']$ then $\Psi(z, s) = [f^{r+q-n'}(z'), r+q, 0]$. Identifying Z_{r+q} with Z, we get a homotopy between f^r and f^{r+q}.

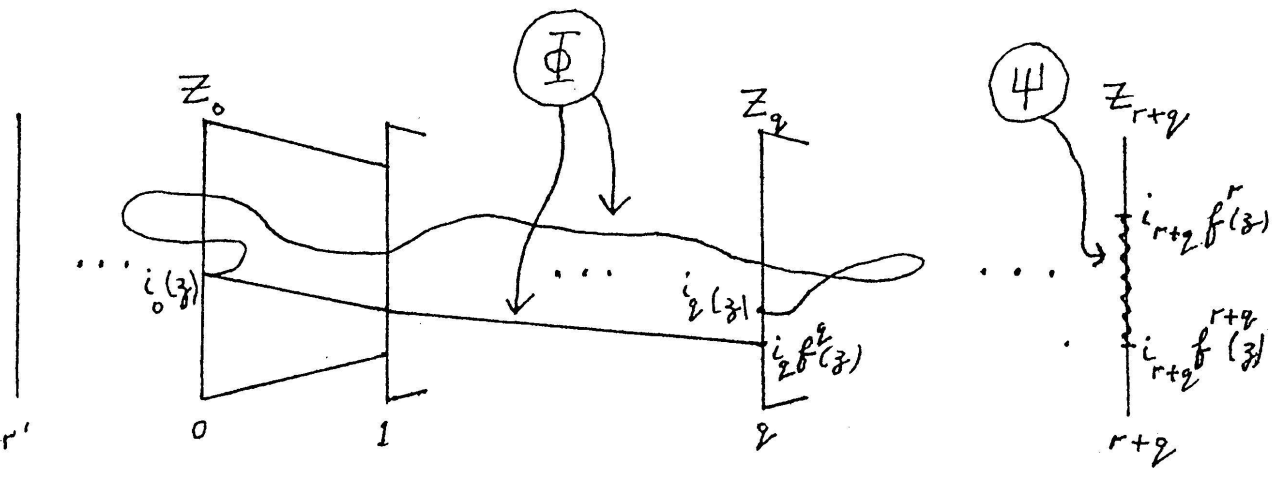

FIGURE 2

It remains to prove eventual coherence. Since $\bar{F}^\gamma$ is $\mathbf{Z}$-equivariant (with respect to the $\mathbf{Z}$-action generated by $\mathscr{T}$), there is a $\mathbf{Z}$-equivariant homotopy $\bar{\Psi} : \bar{X} \times I \to \bar{X}$ such that $\Psi = \bar{\Psi} \circ (i_0 \times \mathrm{id})$.

Consider the diagram:

$$\begin{array}{ccccccc}
Z \times I & \overset{f \times \mathrm{id}}{\longleftarrow} & Z \times I & \overset{\Psi}{\rightarrow} & \bar{X} & \overset{\bar{\bar{f}}}{\rightarrow} & \bar{X} \\
\downarrow i_1 \times \mathrm{id} & & \downarrow i_0 \times \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathscr{E} \\
\bar{X} \times I & = & \bar{X} \times I & \overset{\bar{\Psi}}{\rightarrow} & \bar{X} & = & \bar{X} \\
 & & \uparrow i_1 \times \mathrm{id} & & \uparrow \mathscr{E} & & \\
 & & Z \times I & \overset{\Psi}{\rightarrow} & \bar{X} & &
\end{array}$$

The two middle squares commute. The upper right square commutes up to the homotopy given by:

$$([z, n, u], s) \mapsto \begin{cases} [z, n, u + s] & \text{if } 0 \leqslant s \leqslant 1 - u \\ [f(z), n + 1, u + s - 1) & \text{if } 1 - u \leqslant s \leqslant 1 \,. \end{cases}$$

There is a corresponding homotopy for the upper left square. Thus, the diagram

$$\begin{array}{ccc}
Z \times I & \overset{\Psi}{\rightarrow} & \bar{X} \\
\downarrow f \times \mathrm{id} & & \downarrow \bar{\hat{f}} \\
Z \times I & \overset{\Psi}{\rightarrow} & \bar{X}
\end{array}$$

commutes up to a homotopy $J' : \Psi \circ (f \times \mathrm{id}) \simeq \bar{\hat{f}} \circ \Psi$ which has the property that, for $i = 0$ or 1, the restriction $J'| : Z \times \{i\} \times I \to \bar{X}$ is homotopic rel $Z \times \{i\} \times \{0, 1\}$ to a constant homotopy. Thus, adjusting J', we obtain a homotopy $J'' : Z \times I \times I \to \bar{X}$ rel $Z \times \{0, 1\} \times I$ between $\Psi \circ (f \times \mathrm{id})$ and $\bar{\hat{f}} \circ \Psi$. The argument is finished by "pushing" the track of J'' along the mapping telescope into Z_{r+q+m} where $r + q + m$ is sufficiently large: the details are similar to the construction of Ψ from Φ. We then obtain a homotopy commutative diagram similar to (6.1), showing that f is as claimed. □

Remark. We do not know if every periodic homotopy idempotent $f : Z \to Z$ is eventually coherent. The special case of interest for group theory is the case where Z is aspherical and f is a homotopy equivalence so that we are essentially concerned with an element of the outer automorphism group of $\pi_1(Z)$. A consequence of Proposition 7.3 is that f is indeed eventually coherent in this situation. In the more general case where f is homotopy equivalence but Z is not necessarily aspherical, the obstruction theory of [C] is relevant; see [GN_4].

If (r, N, J, m) are, as above, the data for an eventually coherent periodic homotopy idempotent of period q, we can form $(r, N \star (f^q \circ N), J \star (f^q \circ J), m)$. Here, $N^{(2)} \equiv N \star (f^q \circ N): f^r \simeq f^{r+2q}$, and the concatenation $J^{(2)} \equiv J \star (f^q \circ J)$ takes place in the first I-factor, so that it coincides (after suitable reparametrization) with J on $Z \times \left[0, \frac{1}{2}\right] \times I$ and with $f^q \circ J$ on $Z \times \left[\frac{1}{2}, 1\right] \times I$. One verifies that $(r, N^{(2)}, J^{(2)}, m)$ make Diagram 6.1 commute, hence one has, as above, $S_{(r, N^{(2)}, J^{(2)}, m)} \equiv R_{r+2q+m} \star L^{-1}_{(N^{(2)}, J^{(2)}, m)} \star R^{-1}_{r+m}$, a homotopy from id_X to itself whose rotation degree is $2q$. Iterating this procedure one gets, for any positive integer ν, $S_{(r, N^{(\nu)}, J^{(\nu)}, m)} \equiv R_{r+\nu q+m} \star L^{-1}_{(N^{(\nu)}, J^{(\nu)}, m)} \star R^{-1}_{r+m}$, a homotopy from id_X to itself of rotation degree νq.

PROPOSITION 6.4. *With f and $q > 0$ as in Theorem 6.3, and ν a positive integer, let $\gamma \in \Gamma$ have rotation degree νq. Let (r, N, J, m) be data exhibiting f as an eventually coherent periodic homotopy idempotent of period q. Then there exists $\delta \in \Gamma$ of rotation degree 0 such that $\gamma = \delta[S_{(r, N^{(\nu)}, J^{(\nu)}, m)}]$.*

Proof. Take δ to be $\gamma[S_{(r, N^{(\nu)}, J^{(\nu)}, m)}]^{-1}$. □

Elements of Γ having rotation degree 0 can be "regularized". Let $F^\delta: \mathrm{id}_X \simeq \mathrm{id}_X$ represent such a δ. The basepoint preserving lift is $\bar{F}^\delta: \bar{X} \times I \to \bar{X}$, a homotopy from $\mathrm{id}_{\bar{X}}$ to $\mathrm{id}_{\bar{X}}$. As in the proof of Theorem 6.3, there is an integer $l \geqslant 0$ such that the track, under $\bar{F}^\delta$, of every point $[z, n, u] \in \bar{X}$ can be "pushed" equivariantly into $\{[y, n+l, u] \mid y \in Z\}$. Thus, by an obvious further adjustment, we have:

PROPOSITION 6.5. *If δ has rotation degree 0, then for any sufficiently large l (dependent on δ), F^δ is homotopic rel $X \times \{0, 1\}$ to a homotopy of the form $R_l \star L^{-1}_{(N, J, 0)} \star R_l^{-1}$ where $N: f^l \simeq f^l$ is constructed from F^δ as in the proof of Theorem 6.3.* □

We now prepare to compute the derivation $\tilde{\mathrm{X}}_1(X): \Gamma \to HH_1(\mathbf{Z}G)$.

In the remainder of this section we assume that Z is a finite CW complex, that the map f is cellular and that the basepath σ is cellular. Then $X = T(Z, f)$ inherits a natural CW structure. We will also assume that f is a π_1-*equivalence*; i.e. the induced map $f_\#: \pi_1(Z, v) \to \pi_1(Z, f(v))$ is an isomorphism. Thus $\theta: H \to H$, defined above, is an automorphism. Then the group G is a semidirect product of H with $T \equiv \pi_1(S^1, 1)$; there is an exact sequence: $H \rightarrowtail G \twoheadrightarrow T$ where $H \rightarrowtail G$ is induced by the inclusion $i: Z \hookrightarrow X$

and $G \twoheadrightarrow T$ is induced by p_f. We write $t = [\gamma_0]^{-1} \in G$, projecting to a generator of T, so that $\theta: H \to H$ is given by $h \mapsto tht^{-1}$. We make this choice because we deal with right modules; here and in [GN₂] we prefer "t" rather than "t^{-1}" to appear in our matrices.

Since $\theta: H \to H$ is an isomorphism, the universal cover, $\tilde{X}$, of $X = T(Z, f)$ can be thought of as the mapping telescope of $\tilde{f}: \tilde{Z} \to \tilde{Z}$. Then we have the following model, denoted by $C_*(\tilde{X})$, for the cellular chain complex of $\tilde{X}$. Let $(C_*(\tilde{Z}), {}_Z\tilde{\partial})$ be the cellular chain complex of $\tilde{Z}$. Define $C_*(\tilde{X})$ by

$$C_n(\tilde{X}) = (C_{n-1}(\tilde{Z}) \oplus C_n(\tilde{Z})) \otimes_{\mathbf{Z}} \mathbf{Z}[t, t^{-1}]$$

where the right action of G on $C_n(\tilde{X})$ is given as follows: if $ht^j \in G$ and $a \otimes t^i \in C_n(\tilde{X})$ then $(a \otimes t^i)ht^j \equiv a\theta^i(h) \otimes t^{i+j}$. A choice of oriented lifts of the $(n-1)$-cells and the n-cells of Z determines a finite $\mathbf{Z}G$ basis for the right $\mathbf{Z}G$-module $C_n(\tilde{X})$. The matrix of the boundary operator ${}_X\tilde{\partial}_{n+1}: C_{n+1}(\tilde{X}) \to C_n(\tilde{X})$ with respect to the given $\mathbf{Z}G$ bases is:

$$\begin{bmatrix} [{}_Z\tilde{\partial}_n] & 0 \\ (-1)^{n+1}(I - [\tilde{f}_n]t) & [{}_Z\tilde{\partial}_{n+1}] \end{bmatrix}$$

where $[{}_Z\tilde{\partial}_n]$ is the matrix of ${}_Z\tilde{\partial}_n$, $[\tilde{f}_n]$ is the matrix of $\tilde{f}_n$ and I is an identity matrix of the same size as $[\tilde{f}_n]$. For background on the following calculations, the reader is referred to [GN₂, §4]. See also the Sign Convention in §1.

Let $(\tilde{\mathscr{R}}_k)_n: C_n(\tilde{X}) \to C_{n+1}(\tilde{X})$ be the chain homotopy defined by the k-th tumble R_k. The matrix for $(\tilde{\mathscr{R}}_k)_n$ is:

$$\begin{bmatrix} 0 & (-1)^{n+1} \sum_{i=0}^{k-1} ([\tilde{f}_n]t)^i \\ 0 & 0 \end{bmatrix}.$$

Thus we have:

Proposition 6.6. trace$(\tilde{\partial} \otimes \tilde{\mathscr{R}}_k)$ *is the Hochschild* 1*-chain*

$$\sum_{n \geq 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=0}^{k-1} ([\tilde{f}_n]t)^i).$$

Proof. The identity $d(1 \otimes 1 \otimes g) = 1 \otimes g$ implies that terms of the form trace$(I \otimes M)$ are boundaries and can therefore be ignored. □

Next, suppose f is an eventually coherent periodic homotopy idempotent. As above, we have $r \geq 0$, $N: f^r \simeq f^{r+q}$, $m \geq 0$, and $J: Z \times I \times I \to Z$;

and $L_{(N,J,m)}$ is a homotopy from f^{r+m} to $\tilde{f}^{r+q+m}$. By Proposition 6.2, $L_{(N,J,m)}$ is the concatenation of three homotopies: the first and third of these have zero matrices at the chain homotopy level, and the second, which is $L'_{(N,J,m)}$, is easily seen to give a chain homotopy whose block for $C_n(\tilde{X}) \to C_{n+1}(\tilde{X})$ is

$$\begin{bmatrix} [\tilde{f}^m_n]\,[\tilde{\mathcal{N}}_{n-1}] & 0 \\ W & [\tilde{f}^m_{n+1}]\,[\tilde{\mathcal{N}}_n] \end{bmatrix}.$$

Here, $\tilde{\mathcal{N}}: C_*(\tilde{Z}) \to C_{*+1}(\tilde{Z})$ is the chain homotopy defined by N, and W is a matrix whose exact nature need not concern us. Because of our sign conventions, and the fact that the upper right block is zero we get:

PROPOSITION 6.7. *Let* $\tilde{\mathscr{L}}_{(N,J,m)}: C_*(\tilde{X}) \to C_{*+1}(\tilde{X})$ *be the chain homotopy determined by* $L_{(N,J,m)}$. *Then* $\operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{L}}_{(N,J,m)}) = 0$. □

Now, let $\delta \in \Gamma$ have rotation degree 0. Then $\eta_\#(\delta)$ lies in $H \subset G$ (where η is defined in §1). By Proposition 6.5, we may take $F^\delta = R_l \star L^{-1}_{(N,J,0)} \star R_l^{-1}$ for any sufficiently large l. Under the homotopy $F^\delta: \mathrm{id}_X \simeq \mathrm{id}_X$, the basepoint traverses a loop representing $\eta_\#(\delta)$. Let $\tilde{D}^\delta$ be the chain homotopy defined by $\tilde{F}^\delta$. We rewrite $\eta_\#(\delta) = t^{-l}(t^l\eta_\#(\delta)t^{-l})t^l$. At the matrix level, we then have:

$$\tilde{D}^\delta = \tilde{\mathscr{R}}_l - \tilde{\mathscr{L}}_{(N,J,0)}(t^l\eta_\#(\delta)t^{-l})t^l - \tilde{\mathscr{R}}_l t^{-l}(t^l\eta_\#(\delta)^{-1}).$$

Here, we have used the fact that the matrix of a chain homotopy for a concatenation $A \star B$ is $\mathscr{A} + \mathscr{B}g^{-1}$ where $\mathscr{A}$ and $\mathscr{B}$ are the matrices of A and B and $g \in G$ is the element represented by A(basepoint $\times I$), and the matrix for A^{-1} is $-\mathscr{A}g$. In what follows, recall the right action of Γ on Hochschild chains and homology described in §2. Using Proposition 6.7 we get:

COROLLARY 6.8. *If* $\delta \in \Gamma$ *has rotation degree* 0, *then* $\tilde{\mathrm{X}}_1(X)(\delta)$ *is represented by the Hochschild cycle*

$$\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^\delta) = \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_l)(1 - \delta^{-1})$$

for any sufficiently large l *(dependent on* δ*).* □

Now we return to the situation discussed in Theorem 6.3 and Proposition 6.4. We have $\gamma \in \Gamma$ of rotation degree νq. By Proposition 6.4, $\gamma = \delta[S^{(\nu)}]$ where δ is represented by F^δ, and, for suitably large r and m

(depending only on f), $S^{(\nu)} = R_{r+\nu q+m} \star L^{-1}_{(N^{(\nu)}, J^{(\nu)}, m)} \star R^{-1}_{r+m}$. Under F^{γ}, the basepoint traces out a loop representing

$$\eta_{\#}(\delta)\, t^{-r-\nu q-m}\left(t^{r+\nu q+m}\eta_{\#}(\delta)^{-1}\eta_{\#}(\gamma)\, t^{-r-m}\right) t^{r+m} = \eta_{\#}(\gamma)\,.$$

Here, the four factors correspond to the four parts of the concatenation. Thus

$$\begin{aligned}\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^{\gamma}) &= \operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^{\delta}) + \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+\nu q+m}\eta_{\#}(\delta)^{-1}) \\ &\quad - \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{L}}_{(N,J,m)}t^{r+\nu q+m}\eta_{\#}(\delta)^{-1}) \\ &\quad - \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+m}(\gamma)^{-1})\,.\end{aligned}$$

Using Proposition 6.7 and Corollary 6.8 and the right Γ-action described in Proposition 2.6, this becomes:

$$\begin{aligned}\operatorname{trace}(\tilde{\partial} \otimes \tilde{D}^{\gamma}) &= \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{l})(1-\delta^{-1}) + \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+\nu q+m})\delta^{-1} \\ &\quad - \operatorname{trace}(\tilde{\partial} \otimes \tilde{\mathscr{R}}_{r+m})\gamma^{-1}\,.\end{aligned}$$

In particular, if we enlarge l or $r+m$ so that $l = r+\nu q+m$, and set $\mu = r+m$, we get:

PROPOSITION 6.9. *Let* $\gamma \in \Gamma$ *have rotation degree* $\nu q > 0$ *where* q *is the least positive element of* $P_*(\Gamma)$. *Then* $\tilde{X}_1(X)(\gamma)$ *is represented by the Hochschild cycle:*

$$\sum_{n \geqslant 0} (-1)^n \operatorname{trace}\Big(([\tilde{f}_n]t) \otimes \sum_{i=\mu}^{\mu+\nu q-1} ([\tilde{f}_n]t)^i\Big) + \sum_{n \geqslant 0} (-1)^n \operatorname{trace}\Big(([\tilde{f}_n]t) \otimes \sum_{i=0}^{\mu-1} ([\tilde{f}_n]t)^i\Big)(1-\gamma^{-1})$$

for any sufficiently large positive integer μ *(dependent on* γ*).* □

Remark 6.10. By Corollary 6.8, the same formula holds for γ of rotation degree 0; in that case, the first term in Proposition 6.9 is trivial.

If the subgroup $\Gamma' \subset \Gamma$ is finitely generated by $\gamma_1, \ldots, \gamma_r$ and if the number μ in Proposition 6.9 is taken to be the maximum of the numbers μ_i corresponding to γ_i, then we have an inner derivation $\mathscr{Y}: \Gamma' \to HH_1(\mathbf{Z}G)$ defined at the level of cycles by:

$$\mathscr{Y}(\gamma) = \sum_{n \geqslant 0} (-1)^n \operatorname{trace}\Big(([\tilde{f}_n]t) \otimes \sum_{i=0}^{\mu-1} ([\tilde{f}_n]t)^i\Big)(1-\gamma^{-1})\,.$$

This gives:

COROLLARY 6.11. *If* $i:\Gamma' \hookrightarrow \Gamma$ *is the inclusion of a finitely generated subgroup, there is an inner derivation* $\mathcal{Y}$ *such that for all* $\gamma \in \Gamma'$ *of rotation degree* $\nu q \geqslant 0$, $(\tilde{X}_1(X) - \mathcal{Y})(\gamma)$ *is represented by the Hochschild cycle*

$$\sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=\mu}^{\mu+\nu q-1} ([\tilde{f}_n]t)^i)$$

which therefore depends only on the rotation degree of γ. *In particular, the derivation* $\tilde{X}_1(X) - \mathcal{Y}$ *represents* $i^*(\tilde{\chi}_1(X))$. □

Now we can compute $\chi_1(X):\Gamma \to H_1(X) \equiv G_{ab}$ using Definition A_1.

The automorphism $\theta: H \to H$ induces an automorphism $\theta_{ab}: G_{ab} \to G_{ab}$. We identify G_{ab} with $\operatorname{coker}(\mathrm{id} - \theta_{ab}) \times \mathbf{Z}$ by sending $ht^n \in G$ to $(\{h\}, -n)$. If $\gamma \in \Gamma$ has rotation degree 0, it follows from Corollary 6.8 that $\chi_1(X)(\gamma) = 0$. If $\gamma \in \Gamma$ has rotation degree $\nu q > 0$, we obtain $\chi_1(X)(\gamma)$ in two stages: first apply the augmentation, ε, to the right sides of the tensors in Proposition 6.9, yielding:

$$\sum_{n \geqslant 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=\mu}^{\mu+\nu q-1} [f_n^i]) \in C_1(\mathbf{Z}G, \mathbf{Z})$$

and then apply Proposition 2.1 to get:

$$\sum_{n \geqslant 0} \sum_{i=\mu}^{\mu+\nu q-1} (-1)^n A(\operatorname{trace}([\tilde{f}_n]t[f_n^i]))$$

$$= \sum_{n \geqslant 0} \sum_{i=\mu}^{\mu+\nu q-1} (-1)^n \left[A(\operatorname{trace}([\tilde{f}_n][f_n^i])) + \operatorname{trace}([f_n^i])A(t)\right]$$

which simplifies to:

(6.12)

$$\chi_1(X)(\gamma) = \left(\sum_{n \geqslant 0} \sum_{i=\mu}^{\mu+\nu q-1} (-1)^n A(\operatorname{trace}([\tilde{f}_n][f_n^i])),\ -\sum_{i=\mu}^{\mu+\nu q-1} L(f^i)\right)$$
$$\in \operatorname{coker}(\mathrm{id} - \theta_{ab}) \times \mathbf{Z}.$$

Here, $L(f^i)$ is the Lefschetz number of f^i. Note that the matrix $A([\tilde{f}_n])$ has entries in $\operatorname{coker}(\mathrm{id} - \theta_{ab})$, and for large μ the sequence $(L(f^\mu), \ldots, L(f^{\mu+\nu q-1}))$ is periodic since $f^r \simeq f^{r+q}$.

Summarizing:

THEOREM 6.13. *Let* $f: Z \to Z$ *be a cellular* π_1*-equivalence of a connected CW complex, and let* X *be the mapping torus* $T(Z, f)$.

(i) *if* f *is not an eventually coherent periodic homotopy idempotent, then* $\chi_1(X)(\gamma) = 0$ *for all* $\gamma \in \Gamma$;

(ii) *if* f *is an eventually coherent periodic homotopy idempotent of period* q, *and* $\gamma \in \Gamma$ *has rotation degree* $\nu q > 0$, *the two terms in (6.12) give the two factors of* $\chi_1(X)(\gamma) \in \operatorname{coker}(\mathrm{id} - \theta_{ab}) \times \mathbf{Z}$; *if* γ *has rotation degree* 0, $\chi_1(X)(\gamma) = 0$. □

Remark. If f is not cellular then the above theorem can be applied to any cellular approximation of f. Since any two cellular approximations of f are homotopic, the corresponding mapping tori are homotopy equivalent. By homotopy invariance (Theorem 1.2), this procedure gives a well defined answer.

We get cleaner results when f is also a homotopy equivalence. If, in that case, q is the least positive element of $P_*(\Gamma)$, the proof of Theorem 6.3 shows that f satisfies the eventually coherent periodic homotopy idempotent property with $r = m = 0$; i.e. there is $N\colon \mathrm{id}_Z \simeq f^q$, and J making Diagram 6.1 commute with $m = 0$. The point here is that the inclusions $Z_n \to \bar{X}$ and $\cup_{n=0}^{q-1} M(f)_n \to \bar{X}$ are homotopy equivalences. Since it is now possible to "push" backwards as well as forwards in the telescope $\bar{X}$, we can also take $l = 0$ in the formula preceding Proposition 6.9. Thus we can take $\mu = 0$ in Proposition 6.9:

THEOREM 6.14. *If* f *is a homotopy equivalence and an eventually coherent periodic homotopy idempotent of period* q, *and* $\gamma \in \Gamma$ *has rotation degree* $\nu q > 0$, *then* $\tilde{X}_1(X)(\gamma)$ *is represented by the Hochschild cycle*

$$\sum_{n \geq 0} (-1)^n \operatorname{trace}(([\tilde{f}_n]t) \otimes \sum_{i=0}^{\nu q - 1} ([\tilde{f}_n]t)^i) ;$$

and

$$\chi_1(X)(\gamma) = \left(\sum_{n \geq 0} \sum_{i=0}^{\nu q - 1} (-1)^n A(\operatorname{trace}([\tilde{f}_n][f_n^i])), \; - \sum_{i=0}^{\nu q - 1} L(f^i) \right).$$

These formulas are determined by the rotation degree of γ. □

Example 6.15. Let $f = \mathrm{id}_Z$. Then $X = T(Z, \mathrm{id}_Z) = Z \times S^1$. Let $\nu \geq 0$. The ν-tumble, $\mathscr{R}_\nu$, represents an element of $\Gamma = \pi_1(\mathscr{E}(Z \times S^1), \mathrm{id})$ of rotation degree ν. By Theorem 6.14, we have:

$$\tilde{X}_1(Z \times S^1)([\mathscr{R}_\nu]) = \chi(Z) T_1 \left(t \otimes \frac{1 - t^\nu}{1 - t} \right).$$

This formula also holds for $\nu < 0$. It follows that $\chi_1(Z \times S^1)([\mathscr{R}_\nu]) = (0, -\chi(Z)\nu) = \chi(Z)\nu\{t\}$ where $\{t\} \in H_1(Z \times S^1) \cong H_1(Z) \oplus H_1(S^1)$ is the generator of the $H_1(S^1)$ summand determined by t.

There is a useful simplification of these formulas in the rational case. The identity

$$-\operatorname{trace}(([\tilde{f}_n]t)^{i+1}) \otimes 1 + (i+1)\operatorname{trace}([\tilde{f}_n]t \otimes ([\tilde{f}_n]t)^i)$$
$$= d\left(\sum_{j=1}^{i} \operatorname{trace}([\tilde{f}_n]t \otimes ([\tilde{f}_n]t)^{i-j+1} \otimes ([\tilde{f}_n]t)^{j-1}\right)$$

demonstrates that $\frac{1}{i+1}\operatorname{trace}(([\tilde{f}_n]t)^{i+1}) \otimes 1$ is homologous to $\operatorname{trace}([\tilde{f}_n]t \otimes ([\tilde{f}_n]t)^i)$. We can substitute in Proposition 6.9 and Theorem 6.14. Write $[\tilde{f}]$ for the matrix $\oplus_n(-1)^n[\tilde{f}_n]$. The matrix of the map $\tilde{f}^i$ is $\prod_{j=0}^{i-1}\theta^j([\tilde{f}])$, so $([\tilde{f}]t)^i = [\tilde{f}^i]t^i$. Thus we get:

THEOREM 6.16. *$\tilde{X}_1(X; \mathbf{Q})(\gamma)$ is represented by the Hochschild cycle*

$$\sum_{i=\mu+1}^{\mu+\nu q} \tfrac{1}{i}(\operatorname{trace}[\tilde{f}^i])t^i \otimes 1 + \left(\sum_{i=1}^{\mu} \tfrac{1}{i}(\operatorname{trace}[\tilde{f}^i])t^i \otimes 1\right)(1-\gamma^{-1})$$

for any sufficiently large positive integer μ (dependent on γ). When f is also a homotopy equivalence then $\tilde{X}_1(X; \mathbf{Q})(\gamma)$ is represented by the Hochschild cycle

$$\sum_{i=1}^{\nu q} \tfrac{1}{i}(\operatorname{trace}[\tilde{f}^i])t^i \otimes 1 . \quad \square$$

Remark. The formula for $\tilde{X}_1(X; \mathbf{Q})(\gamma)$ above can be expressed in terms of the "reduced Reidemeister traces" of the iterates f^n, $n = 1, \ldots, \nu q$. This trace of f^n take values in the "reduced" 0-th Hochschild homology group of $\mathbf{Z}H$ with θ^n-twisted coefficients; see [GN$_2$, §5].

The computation of $\chi_1(X; \mathbf{Q})$ naturally leads one to consider the *homology Reidemeister trace* of a cellular map $f: Z \to Z$, denoted by $L^h(f)$. It is the element of $H_1(H) \cong H_{ab}$ given by

$$L^h(f) = \sum_{n \geq 0} (-1)^n A(\operatorname{trace}([\tilde{f}_n])) .$$

If k is a commutative ring of coefficients, let $L^h(f; k)$ denote the image of $L^h(f)$ under the homomorphism $H_1(H) \to H_1(H; k)$. Let $\bar{L}^h(f; k) \in \operatorname{coker}(\mathrm{id} - \theta_{ab} \otimes \mathrm{id}_k)$ denote the image of $L^h(f; k)$. It is easy to see that $L^h(f)$ and $\bar{L}^h(f; k)$ depend only on the homotopy class of f.

(Both $\bar{L}^h(f; k)$ and $L^h(f; k)$ have an interpretation in terms of Nielsen fixed point theory, but we will not make use of this.)

Theorem 6.16 together with the proof of (6.12) yields the following formula. For all sufficiently large μ:

$$\chi_1(X; \mathbf{Q})(\gamma) = \sum_{i=\mu+1}^{\mu+\nu q} \left(\tfrac{1}{i}\bar{L}^h(f^i; \mathbf{Q}), -L(f^i)\right).$$

Since this formula is valid for *all* sufficiently large μ, it is easy to see (because of periodicity and the appearance of the coefficients $\frac{1}{i}$) that:

COROLLARY 6.17. *For all sufficiently large* i, $\bar{L}^h(f^i; \mathbf{Q}) = 0$. □

Thus:

COROLLARY 6.18. *For all sufficiently large* μ:

$$\chi_1(X; \mathbf{Q})(\gamma) = \left(0, -\sum_{i=\mu+1}^{\mu+\nu q} L(f^i)\right).$$

In particular, if f *is also homotopy equivalence*

$$\chi_1(X; \mathbf{Q})(\gamma) = \left(0, -\sum_{i=0}^{\nu q-1} L(f^i)\right). \quad \square$$

7. MORE ON GROUPS OF TYPE $\mathscr{F}$

We consider in more detail the special case of the mapping torus of a homotopy equivalence of an aspherical complex.

Let H be an arbitrary group, let $\theta: H \to H$ be an automorphism, and let G be the semidirect product $\langle H, t \mid tht^{-1} = \theta(h)$ for all $h \in H\rangle$. Write $\mathrm{Fix}(\theta) = \{h \in H \mid \theta(h) = h\}$ and write $\langle x\rangle$ for the cyclic subgroup generated by $x \in G$. Let $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$ be the group of outer automorphisms of H, i.e. the quotient of the group, $\mathrm{Aut}(H)$, of automorphisms of H by the normal subgroup $\mathrm{Inn}(H)$ of inner automorphisms.

LEMMA 7.1. *If* θ *has infinite order in* $\mathrm{Out}(H)$, *then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$. *If* θ *has finite order* r *in* $\mathrm{Out}(H)$, *and* $h_0 \in H$ *is such that* $\theta^r(\cdot) = h_0(\cdot)h_0^{-1}$, *there are two cases:*

(1) *No positive power of* h_0 *lies in* $Z(H)\mathrm{Fix}(\theta)$. *Then* $Z(G) = Z(H) \cap \mathrm{Fix}(\theta)$.