1.2 Homotopy types

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The main classification result can now be formulated in the following way:

THEOREM 1 (Jupp). *The assignment*

$$X \mapsto \left(\tfrac{b_3(X)}{2}, H^2(X, \mathbf{Z}), w_2(X), \tau(X), F_X, p_1(X)\right)$$

induces a 1-1 correspondence between oriented homeomorphism classes of 1-connected, closed, oriented, 6-dimensional topological manifolds with torsion-free homology, and equivalence classes of admissible systems of invariants.

Furthermore, a topological manifold X *as above admits a* C^∞*-structure if and only if the triangulation class* $\tau(X)$ *vanishes; the* C^∞*-structure is then unique.*

REMARK 1. The classification theorem is due to C.T.C. Wall in the special case of differentiable spin-manifolds [W]; the final form above was obtained by P. Jupp [J].

A. Žubr generalized Wall's result in another direction; he proved a classification theorem for 1-connected, smooth spin-manifolds with not necessarily torsion-free homology [Z1]; in two further papers [Z2], [Z3] he also obtains P. Jupp's classification, and he asserts in addition, that algebraic isomorphisms of systems of invariants can always be realized by orientation preserving homeomorphisms (diffeomorphisms in the smooth case).

Note that the first invariant $\frac{b_3(X)}{2}$ of the system is completely independent of the remaining invariants, so that the following splitting theorem holds:

COROLLARY 1. *Every 1-connected, closed, oriented, 6-dimensional, topological (differentiable) manifold* X *with torsion-free homology admits a topological (differentiable) splitting* $X = X_0 \# \frac{b_3(X)}{2}(S^3 \times S^3)$ *as a connected sum of a core* X_0 *with* $b_3(X_0) = 0$, *and* $\frac{b_3(X)}{2}$ *copies of* $S^3 \times S^3$. *The oriented homeomorphism (diffeomorphism) type of* X_0 *is unique.*

EXAMPLE 1. The 1-connected, closed, oriented 6-manifolds X with $H_2(X, \mathbf{Z}) = 0$ are S^6 and the connected sums $\#_r S^3 \times S^3$ of $r \geqslant 1$ copies of $S^3 \times S^3$ [*Sm*].

1.2 HOMOTOPY TYPES

In order to describe the homotopy classification of the 6-manifolds above, we need some more preparations.

Let (H, F) be a pair consisting of a finitely generated free abelian group H, and a symmetric trilinear form F; consider the following subgroup of $H^\vee/_{48H^\vee}$:

$$U_F := \{l \in H^\vee/_{48H^\vee} \mid \exists u \in H \text{ with } l(x) \equiv 24u^2 \cdot x (\bmod 48)\ \forall x \in H\} .$$

If (H', F') is another such pair, and $\alpha : H \to H'$ an isomorphism with $\alpha^*(F') = F$, then there is an induced isomorphism

$$\alpha^* : H'^\vee/_{48H'^\vee}/_{U'_F} \to H^\vee/_{48H^\vee}/_{U_F}$$

of the quotients. Denote the class of a linear form $l \in H^\vee$ in the quotient $H^\vee/_{48H^\vee}/_{U_F}$ by $[l]$.

DEFINITION 2. *Two systems of invariants* (r, H, w, τ, F, p) *and* $(r', H', w', \tau', F', p')$ *are weakly equivalent iff* $r = r'$, *and there exists an isomorphism* $\alpha : H \to H'$ *such that:*

$$\alpha(w) = w', \alpha^*(F') = F, \text{ and } \alpha^*[p' + 24T'] = [p + 24T]$$
$$\text{for all } T \in H^\vee, T' \in H'^\vee \text{ with } \bar{T} \equiv \tau (\bmod 2), \bar{T}' \equiv \tau' (\bmod 2) .$$

With this definition we can phrase the homotopy classification in the following way:

THEOREM 2 (Žubr). *The assignment*

$$X \to \left(\tfrac{b_3(X)}{2}, H^2(X, \mathbf{Z}), w_2(X), \tau(X), F_X, p_1(X)\right)$$

induces a 1-1 correspondence between oriented homotopy classes of 1-connected, closed, oriented, 6-dimensional topological manifolds with torsion-free homology and weak equivalence classes of admissible systems of invariants.

REMARK 2. Žubr's theorem corrects and generalizes the homotopy classification in the papers by Wall [W] and Jupp [J]; he also treats manifolds with not necessarily torsion-free homology, and states without proof that algebraic isomorphisms of weak equivalence classes of systems of invariants are always realizable by orientation preserving homotopy equivalences [Z3].

EXAMPLE 2. Manifolds with $b_2(X) = 1$.

Let X be a 1-connected, closed, oriented, 6-dimensional manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$. Splitting off possible copies of $S^3 \times S^3$ we may assume $b_3(X) = 0$. Choosing a $\mathbf{Z}$-basis of $H^2(X, \mathbf{Z})$ we see that systems of invariants can be identified with 4-tuples $(\bar{W}, \bar{T}, d, p) \in \mathbf{Z}_{/2} \times \mathbf{Z}_{/2} \times \mathbf{Z} \times \mathbf{Z}$

where the 'degree' d corresponds to the cubic form. Such a 4-tuple is admissible iff $d(2x + \bar{W})^3 \equiv (p + 24\bar{T}) \cdot (2x + \bar{W}) \pmod{48}$ holds for every integer x. This is equivalent to $p \equiv 4d \pmod{24}$ if $\bar{W} = 0$, and to $p \equiv d + 24\bar{T} \pmod{48}$ with $d \equiv 0 \pmod 2$ if $\bar{W} \neq 0$.

Two admissible 4-tuples $(\bar{W}, \bar{T}, d, p)$ and $(\bar{W}', \bar{T}', d', p')$ are equivalent iff $\bar{W}' = \bar{W}, \bar{T}' = \bar{T}$ and $(d', p') = \pm (d, p)$. Taking the degree d non-negative, we find:

PROPOSITION 1. *There is a 1-1 correspondence between oriented homeomorphism types of cores* X_0 *with* $b_2(X_0) = 1$, *and 4-tuples* $(\bar{W}, \bar{T}, d, p)$, *normalized so that* $d \geqslant 0$, *and* $p \geqslant 0$ *if* $d = 0$, *which satisfy* $p \equiv 4d \pmod{24}$ *if* $\bar{W} = 0$, *and* $d \equiv 0 \pmod 2$, $p \equiv d + 24\bar{T} \pmod{48}$ *if* $\bar{W} \neq 0$.

In order to classify the associated homotopy types we first have to determine the subgroup U_F associated to a given cubic form F. By definition we find $U_F = 0$ if $d \equiv 0 \pmod 2$, $U_F = \mathbf{Z}_{/2}$ if $d \equiv 1 \pmod 2$. Two normalized 4-tuples $(\bar{W}, \bar{T}, d, p)$ and $(\bar{W}', \bar{T}', d', p')$ are weakly equivalent iff $d' = d, \bar{W}' = \bar{W}$, and $p + 24\bar{T} \equiv p' + 24\bar{T}' \pmod{48}$ if $d \equiv 0 \pmod 2$, $p \equiv p' \pmod{24}$ if $d \equiv 1 \pmod 2$.

Putting everything together, we find a single oriented homotopy type for every odd degree $d \geqslant 0$, which is necessarily spin, and 3 oriented homotopy types for every even degree $d \geqslant 0$; one of these 3 types has $\bar{W} \neq 0$, the other two are spin, and they are distinguished by $p + 24\bar{T} \pmod{48}$ i.e. $p \equiv 4d \pmod{48}$, or $p \equiv 4d + 24 \pmod{48}$.

2. REALIZATION OF CUBIC FORMS

In the previous section the (homotopy) topological classification of 1-connected, closed, oriented, 6-dimensional manifolds with torsion-free homology has been transformed into an arithmetical moduli problem: to describe the sets of (weak) equivalence classes of admissible systems of invariants. In this section we begin to investigate the latter problem; we give a simple criterion for the realizability of cubic forms by smooth manifolds, and we describe, at least in principle, the classification of homotopy types of manifolds with a given cohomology ring.