3. Algebra and arithmetic of cubic forms

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3. ALGEBRA AND ARITHMETIC OF CUBIC FORMS

Let H be a finitely generated free $\mathbf{Z}$-module of rank b. In this section we want to study algebraic and arithmetic properties of symmetric trilinear forms $F \in S^3H^\vee$ on H which admit characteristic elements; ultimately we would like to describe the classification of those forms under the action of the general linear group $GL(H)$, i.e. we like to investigate (part of) the quotient $S^3H^\vee/_{GL(H)}$.

From what we have said in sections 1 and 2, this is clearly equivalent to classifying the cohomology rings of 1-connected, closed, oriented, 6-dimensional manifolds without torsion, and with $b_2 = b$, $b_3 = 0$. Furthermore, up to finite indeterminancy, this is also equivalent to classifying the homotopy types of these manifolds.

The proper setting for this arithmetic moduli problem can be found in C. Seshadri's paper [S]; here we investigate only its set-theoretic aspects. Let $H_{\mathbf{C}} := H \otimes_{\mathbf{Z}} \mathbf{C}$ be the complexification of H, and let $S^3H^\vee_{\mathbf{C}}/_{SL(H_{\mathbf{C}})}$ be the quotient of the reductive group $SL(H_{\mathbf{C}})$. We obtain a natural map $c: S^3H^\vee/_{SL(H)} \to S^3H^\vee_{\mathbf{C}}/_{SL(H_{\mathbf{C}})}$, which allows us to break up the problem into three parts: the description of the quotient $S^3H^\vee_{\mathbf{C}}/_{SL(H_{\mathbf{C}})}$, the investigation of the fibers of c, and the study of the remaining $\mathbf{Z}_{/2}$-action on $S^3H^\vee/_{SL(H)}$ which is induced by the choice of an arbitrary automorphism $A_\circ \in GL(H)$ of determinant $\det A_\circ = -1$.

3.1 ALGEBRAIC PROPERTIES OF CUBIC FORMS

Let $H_{\mathbf{C}} = H \otimes_{\mathbf{Z}} \mathbf{C}$ be as above, and denote by $\mathbf{C}[H_{\mathbf{C}}]_3$ the space of homogeneous polynomials of degree 3 on $H_{\mathbf{C}}$. There exists a linear polarization operator $\mathrm{Pol}: \mathbf{C}[H_{\mathbf{C}}]_3 \to S^3H^\vee_{\mathbf{C}}$, sending a homogeneous cubic polynomial $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ to the symmetric trilinear form $F = \mathrm{Pol}(f) \in S^3H^\vee_{\mathbf{C}}$ which is related to f by the identity $F(h, h, h) = 6f(h)$. We will usually not distinguish between a cubic polynomial f and its associated form $F = \mathrm{Pol}(f)$. On $S^3H^\vee_{\mathbf{C}}$ there exists a polynomial function $\Delta: S^3H^\vee_{\mathbf{C}} \to \mathbf{C}$, the discriminant, which is homogeneous of degree $b \cdot 2^{b-1}$, and vanishes in a form F if and only if the associated cubic hypersurface $(f)_\circ \subset \mathbf{P}(H_{\mathbf{C}})$ has a singular point; Δ is defined over $\mathbf{Z}$ and is clearly invariant under the natural action of $SL(H_{\mathbf{C}})$.

REMARK 4. Of course, a discriminant function Δ exists for forms of arbitrary degree d; in the general case Δ is homogeneous of degree $b \cdot (d-1)^{b-1}$ on $S^dH^\vee_{\mathbf{C}}$.

PROPOSITION 5. *Fix a symmetric trilinear form* $F \in S^3 H^\vee_{\mathbf{C}}$ *and an element* $h \in H_{\mathbf{C}} \setminus \{0\}$ *with* $f(h) = 0$. *The associated point* $<h> \in \mathbf{P}(H_{\mathbf{C}})$ *is a singular point of the cubic hypersurface* $(f)_0 \subset \mathbf{P}(H_{\mathbf{C}})$ *if and only if the linear form* $h^2 \in H^\vee_{\mathbf{C}}$ *is zero. The existence of at least one such point is equivalent to the vanishing of the discriminant.*

Proof. From $f(h + tv) = f(h) + 3th^2 \cdot v + 3t^2 h \cdot v^2 + t^3 v^3$ for every $v \in H_{\mathbf{C}}, t \in \mathbf{C}$ we find $\frac{d}{dt}|_0 f(h + tv) = 3h^2 \cdot v$, i.e. $h^2 \in H^\vee_{\mathbf{C}}$ defines the differential of f in h.

REMARK 5. $\mathbf{Q}$-rational points in $(f)_0 \subset \mathbf{P}(H_{\mathbf{C}})$, and $\mathbf{Q}$-rational singularities of $(f)_0$ have geometric significance if the cubic f is defined by the cup-form of a 6-manifold X. In fact, integral classes $h \in H^2(X, \mathbf{Z})$ correspond to homotopy classes of maps to $\mathbf{P}^3_{\mathbf{C}}$; such a map factors over $\mathbf{P}^2_{\mathbf{C}} \subset \mathbf{P}^3_{\mathbf{C}}$ if and only if $h^3 = 0$; if it factors over $\mathbf{P}^1_{\mathbf{C}} \subset \mathbf{P}^3_{\mathbf{C}}$, then clearly $h^2 = 0$. The converse will probably not always be true since, in general, the cohomology ring does not determine the homotopy type.

In addition to the invariant discriminant $\Delta(f)$ of a polynomial f, we will also need a fundamental covariant H_f, the Hessian of f. Let $F = \mathrm{Pol}(f) \in S^3 H^\vee_{\mathbf{C}}$ be the polarization of $f \in \mathbf{C}[H_{\mathbf{C}}]_3$; the Hessian of f can then be defined as the composition $H_f: H_{\mathbf{C}} \xrightarrow{F^t} S^2 H^\vee_{\mathbf{C}} \xrightarrow{\mathrm{disc}} \mathbf{C}$, i.e. H_f is the homogeneous polynomial function of degree b on $H_{\mathbf{C}}$ given by $H_f(h) = \mathrm{disc}(F^t(h))$. In terms of linear coordinates $\xi_1, \cdots, \xi_b$ on H one finds the more familiar expression $H_f = \det\left(\frac{\partial^2}{\partial \xi_i \partial \xi_j} f\right)$.

PROPOSITION 6. *Let* $F \in S^3 H^\vee_{\mathbf{C}}$ *be a symmetric trilinear form. The Hessian of* F *is identically zero if and only if there exists no element* $h \in H_{\mathbf{C}}$ *for which the map* $\cdot h: H_{\mathbf{C}} \to H^\vee_{\mathbf{C}}$ *is an isomorphism.*

Proof. H_f is identically zero if and only if the symmetric bilinear forms $F^t(h) \in S^2 H^\vee_{\mathbf{C}}$ are degenerate for every $h \in H_{\mathbf{C}}$. But this means that none of the maps $\cdot h: H_{\mathbf{C}} \to H^\vee_{\mathbf{C}}$ is an isomorphism.

COROLLARY 3. *Let* $F \in S^3 H^\vee_{\mathbf{C}}$ *be a form whose associated map* $F^t: H_{\mathbf{C}} \to S^2 H^\vee_{\mathbf{C}}$ *is not injective. Then we have* $H_f = 0$.

Proof. Let $k \in \mathrm{Ker}(F^t)$ be a non-zero element, and consider an arbitrary element $h \in H_{\mathbf{C}}$. By definition of k we have $F(k, h, v) = 0$ for all $v \in H_{\mathbf{C}}$, i.e. $k \cdot h \in H^\vee_{\mathbf{C}}$ is zero.

REMARK 6. It is not difficult to show that F^t is not injective if and only if there exists a proper quotient $\bar{H}_{\mathbf{C}}$ of $H_{\mathbf{C}}$, and a form $\bar{F} \in S^3 \bar{H}_{\mathbf{C}}^\vee$ whose pull-back to $H_{\mathbf{C}}$ is the given form F. This means that the Hessians of cubic polynomials $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ which 'do not depend on all variables' are automatically zero.

The converse holds for forms in $b \leqslant 4$ variables, but not in general [G/N].

3.2 THE GIT QUOTIENT $S^3 H_{\mathbf{C}}^\vee /\!/_{SL(H_{\mathbf{C}})}$

Let $V := S^3 H_{\mathbf{C}}^\vee$ be the vector space of complex cubic forms. The reductive group $G := SL(H_{\mathbf{C}})$ acts rationally on V, and therefore has a finitely generated ring $\mathbf{C}[V]^G$ of invariants [H]. The inclusion $\mathbf{C}[V]^G \subset \mathbf{C}[V]$ induces a regular map $\pi: V \to V/\!/_G$ onto the affine variety $V/\!/_G$ with coordinate ring $\mathbf{C}[V]^G$. It is well known that π is a categorical quotient, which is G-closed and G-separating, so that $V/\!/_G$ parametrizes precisely the closed G-orbits in V. Recall that a point $v \in V$ is semi-stable if $o \notin \overline{G \cdot v}$, and that v is stable if $G \cdot v$ is closed in V and the isotropy group G_v is finite [M/F]. Denote the G-invariant, open subsets of semistable (stable) points in V by $V^{ss}(V^s)$.

The complement $V \setminus V^{ss} = \pi^{-1}(\pi(0))$ consists of 'Nullformen', i.e. forms for which all polynomial invariants vanish. The open subset of stable points, which includes in particular all non-singular forms, has a geometric quotient, given by the restricted map $\pi \mid V^s : V^s \to \pi(V^s)$.

REMARK 7. Let $A_\circ \in GL(H)$ be a fixed automorphism of determinant $\det A_\circ = -1$, e.g. $A_\circ = -id_H$ if b is odd. $A_\circ$ induces a $\mathbf{Z}_{/2}$-action on $S^3 H^\vee /_{SL(H)}$ and on $S^3 H_{\mathbf{C}}^\vee /_{SL(H_{\mathbf{C}})}$, for which the map c is equivariant.

Let $\hat{G} \subset GL(H_{\mathbf{C}})$ be the semi-direct product of $SL(H_{\mathbf{C}})$ and $\mathbf{Z}_{/2}$ generated by $A_\circ$ and $SL(H_{\mathbf{C}})$. The invariant ring $\mathbf{C}[V]^{\hat{G}}$ has an important topological interpretation: it consists of all polynomial invariants of complex cohomology rings of 1-connected, closed, oriented 6-dimensional manifolds with torsion-free homology.

EXAMPLE 5. Binary cubics ($b = 2$)

Choose linear coordinates X, Y on $H_{\mathbf{C}}$, and write a cubic polynomial $f \in \mathbf{C}[X, Y]_3$ in the form $f = a_0 X^3 + 3a_1 X^2 Y + 3a_2 XY^2 + a_3 Y^3$.

We use a_0, a_1, a_2, a_3 as coordinates on $S^3 H_{\mathbf{C}}^\vee$, so that $\mathbf{C}[S^3 H_{\mathbf{C}}^\vee] = \mathbf{C}[a_0, a_1, a_2, a_3]$. The discriminant $\Delta(f)$ of f is a homogeneous polynomial of degree 4 in the coefficients a_0, a_1, a_2, a_4, explicitly given by $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$.

The discriminant generates the ring of $SL(H_\mathbf{C})$-invariants,

$$\mathbf{C}[S^3 H_\mathbf{C}^\vee]^{SL(H_\mathbf{C})} = \mathbf{C}[\Delta] ,$$

and it is easy to see that Δ is also $\mathbf{Z}_{/2}$-invariant. A cubic form f is stable if and only if it is semistable, if and only if it is non-singular [N]. The cone of nullforms $\pi^{-1}(\pi(0))$ is the affine hypersurface $(\Delta)_0 \subset S^3 H_\mathbf{C}^\vee$; it has a nice geometric interpretation in terms of the Hessian. The Hessian of the cubic f is the quadratic form

$$H_f = 6^2[(a_0 a_2 - a_1^2)X^2 + (a_0 a_3 - a_1 a_2)XY + (a_1 a_3 - a_2^2)Y^2] .$$

The set of forms f with vanishing Hessians H_f form the affine cone over the rational normal curve in $\mathbf{P}(S^3 H_\mathbf{C}^\vee)$; the hypersurface of nullforms is the cone over the tangential scroll of this curve. There are 4 different types of $SL(H_\mathbf{C})$-orbits in $S^3 H_\mathbf{C}^\vee$, represented by the normal forms $XY(X + \lambda Y)$, $X^2 Y$, X^3, 0. The first type is stable, the others are nullforms, the orbits of X^3 and 0 have vanishing Hessians.

EXAMPLE 6. Ternary cubics ($b = 3$)

The ring of $SL(H_\mathbf{C})$-invariants of ternary cubics is a weighted polynomial ring in 2 variables, $\mathbf{C}[S^3 H_\mathbf{C}^\vee]^{SL(H_\mathbf{C})} = \mathbf{C}[S, T]$ whose generators S, T have been found by S. Aronhold [A]. S is a homogeneous polynomial of degree 4 in the coefficients of a cubic f, T is homogeneous of degree 6, both polynomials are $\mathbf{Z}_{/2}$-invariant. For a cubic of the form $f = aX^3 + bY^3 + cZ^3 + 6dXYZ$, S and T are given by $S = 4d(d^3 - abc)$ and $T = 8d^6 + 20abc(d^3 - abc)$ respectively [P]. The general formulae, which take two pages to write down, can be found in the book of Sturmfels [St]. The discriminant of a form f is homogeneous of degree 12 in the coefficients of f; in terms of Aronhold's invariants S, T it is simply given by $\Delta = S^3 - T^2$. We obtain the following overall picture: The GIT quotient for ternary cubics is an affine plane $\mathbf{A}^2$ with coordinates S, T. The complement $\mathbf{A}^2 \setminus (\Delta)_0$ of the discriminant curve is the geometric quotient of stable cubics. The π-fibers over a point $(S, T) \neq (0, 0)$ on the discriminant curve $(\Delta)_0$ consist of 3 types of $SL(H_\mathbf{C})$-orbits: nodal cubics with normal form $X^3 + Y^3 + 6\alpha XYZ$, reducible cubics formed by a smooth conic and a transversal line (normal form: $X^3 + 6\alpha XYZ$), and cubics consisting of three lines in general position (normal form: $6\alpha XYZ$); these cubics are properly semi-stable for $\alpha \neq 0$ with Aronhold invariants $S = 4\alpha^4$, $T = 8\alpha^6$. The fiber of π over 0 contains 6 orbits with normal forms

$$Y^2Z - X^3, Y(X^2 - YZ), XY(X+Y), X^2Y, X^3,$$

and 0, of which the last 4 types have vanishing Hessians. For more details we refer to H. Kraft's book [Kr].

REMARK 8. The natural $\mathbf{C}^*$-action $f \to \lambda \cdot f$ on cubic forms induces a weighted action on the GIT quotient $S^3H^\vee_{\mathbf{C}}/_{SL(H_{\mathbf{C}})}$, $\lambda \cdot (S, T) = (\lambda^4 S, \lambda^6 T)$. The associated weighted projective space $\mathbf{P}^1(4, 6)$ with homogeneous coordinates $<S, T>$ is the good quotient for semi-stable plane cubic curves. Its affine part $\mathbf{P}^1 \backslash (\Delta)_0$ is the moduli space of genus-1 curves. The $PGL(H_{\mathbf{C}})$-invariant $J := \frac{S3}{\Delta}$ gives the J-invariant of the corresponding curve.

3.3 ARITHMETICAL ASPECTS

Let $c: S^3H^\vee/_{SL(H)} \to S^3H^\vee_{\mathbf{C}}/_{SL(H)}$ be the map which associates to the $SL(H)$-orbit $<F>$ of a symmetric trilinear form $F \in S^3H^\vee$ the $SL(H_{\mathbf{C}})$-orbit $<F>_{\mathbf{C}}$ of its complexification. The c-fiber over $<F>_{\mathbf{C}}$ can be identified with the subset $(SL(H_{\mathbf{C}}) \cdot F \cap S^3H^\vee)/_{SL(H)}$ of $S^3H^\vee/_{SL(H)}$. C. Jordan has shown that these subsets are finite provided the cubic form $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ associated to F has a non-vanishing discriminant [J1]. Jordan's original proof, which is only two pages long, is somewhat hard to follow. The following theorem of A. Borel and Harish-Chandra provides, however, a vast generalization of Jordan's finiteness result:

THEOREM 3 (Borel/Harish-Chandra). *Let G be a reductive $\mathbf{Q}$-group, $\Gamma \subset G$ an arithmetic subgroup, $\xi: G \to GL(V)$ a $\mathbf{Q}$-morphism, and $L \subset V$ a Γ-invariant sublattice of $V_{\mathbf{Q}}$. If $v \in V$ has a closed G-orbit in V, then $G_v \cap L/_\Gamma$ is a finite set.*

Proof. [B].

COROLLARY 4. *Let $F \in S^3H^\vee$ be a symmetric trilinear form on H. If the $SL(H_{\mathbf{C}})$-orbit of F in $S^3H^\vee_{\mathbf{C}}$ is closed, then the fiber $c^{-1}(<F>_{\mathbf{C}})$ over $<F>_{\mathbf{C}}$ is finite.*

To check whether a $SL(H_{\mathbf{C}})$-orbit $SL(H_{\mathbf{C}}) \cdot F$ is closed in $S^3H^\vee_{\mathbf{C}}$, one has a generalization of the Hilbert-criterion [Kr]: $SL(H_{\mathbf{C}}) \cdot F$ is closed in $S^3H^\vee_{\mathbf{C}}$ if and only if for every 1-parameter subgroup $\lambda: \mathbf{C}^* \to SL(H_{\mathbf{C}})$, for which $lim_{t\to 0}\lambda(t)\cdot F$ exists in $S^3H^\vee_{\mathbf{C}}$, this limit is already contained in $SL(H_{\mathbf{C}})\cdot F$. A sufficient condition for $SL(H_{\mathbf{C}})\cdot F$ to be closed follows from another result of C. Jordan [J2]:

THEOREM 4 (Jordan). *Let* $f\in \mathbf{C}[H_{\mathbf{C}}]_d$ *be a homogeneous polynomial of degree* $d\geqslant 3$. *If its discriminant* $\Delta(f)$ *is non-zero, then* f *has a finite isotropy group* $SL(H_{\mathbf{C}})_f$.

COROLLARY 5. *Let* $F\in S^3H^\vee$ *be a form whose associated cubic polynomial* $f\in\mathbf{C}[H_{\mathbf{C}}]_3$ *has* $\Delta(f)\neq 0$. *Then* $SL(H_{\mathbf{C}})\cdot F$ *is closed in* $S^3H^\vee_{\mathbf{C}}$.

Proof. Standard arguments, cf. [Bo].

REMARK 9. Closedness of the $SL(H_{\mathbf{C}})$-orbit of F is only a sufficient condition for the finiteness of the fiber $c^{-1}(<F>_{\mathbf{C}})$. There exist other finiteness theorems for special types of forms, like e.g. forms which decompose into linear factors.

Some of these results are surveyed in Volume III of L. Dickson's book [D].

We say that two forms $F, F'\in S^3H^\vee$ belong to the same (proper) equivalence class if they lie in the same $(SL(H)\text{-})GL(H)$-orbit. The group $\mathbf{Z}_{/2}=GL(H)/_{SL(H)}$ acts on the set $S^3H^\vee/_{SL(H)}$ of proper classes, and the quotient becomes the orbit space $S^3H^\vee/_{GL(H)}$.

The $\mathbf{Z}_{/2}$-action is not free in general, but for finiteness properties this plays no rôle.

EXAMPLE 7. Binary cubics

Let H be a free $\mathbf{Z}$-module of rank $b=2$. There exist only finitely many classes of symmetric trilinear forms $F\in S^3H^\vee$ with a given non-zero discriminant Δ. Of course, Δ must be integral, and a square modulo 4, in order to be realizable by an integral form. For some small values of $\Delta\neq 0$ the number of classes is known. Results in this direction go back to a paper by F. Arndt [A]; his tables have been rearranged by A. Cayley [Cay]. It should certainly be possible to go much further using modern computers.

EXAMPLE 8. Ternary cubics

Let H be a free $\mathbf{Z}$-module of rank 3 with coordinates X, Y, Z. The cubic polynomials with closed $SL(H_{\mathbf{C}})$-orbits are the non-singular cubics, and the polynomials in the orbits of $6\alpha XYZ$ for all $\alpha\in\mathbf{C}$.

The number of integral classes in these orbits is therefore finite. We have, however, an even stronger finiteness theorem for stable ternary cubics:

PROPOSITION 7. *Let H be a free $\mathbf{Z}$-module of rank 3. There exist only finitely many classes of symmetric trilinear forms $F \in S^3 H^\vee$ with a fixed discriminant $\Delta \neq 0$.*

Proof. In terms of Arnhold's invariants S and T, Δ is given by $\Delta = S^3 - T^2$. By a theorem of C. Siegel [Si], the diophantine equation $S^3 - T^2 = \Delta$ has only finitely many integral solution (S, T) for any integer $\Delta \neq 0$. For each of these solutions the corresponding point in $S^3 H_{\mathbf{C}}^\vee / _{SL(H_{\mathbf{C}})}$ lies outside of the discriminant curve, so that the π-fiber over it is a closed $SL(H_{\mathbf{C}})$-orbit. The finiteness of the class number then follows from the Borel/Harish-Chandra theorem.

A famous special case of Siegel's theorem is Bachet's equation $S^3 - T^2 = 2$; it has only the two obvious solutions $(3, \pm 5)$.

REMARK 10. To get finiteness results for ternary cubic forms it is not sufficient to fix the J-invariant (instead of the discriminant): The forms $f_m = X^3 + XZ^2 + Z^3 + mY^2Z$, $m \in \mathbf{Z} \setminus \{0\}$, all have the same J-invariant, but they are not equivalent, even over $\mathbf{Q}$, since they have bad reduction at different primes $p \mid m$.

4. INVARIANTS OF COMPLEX 3-FOLDS

In this section we begin to investigate the topology of 1-connected, compact, complex 3-folds. After a brief discussion of the possible systems of Chern numbers of almost complex 6-manifolds, we study the behaviour of the topological invariants of complex 3-folds under certain standard constructions, like e.g. branched coverings, or blow-ups of points and curves. Then we describe some interesting examples of 1-connected, non-Kählerian 3-folds, including a new construction method which generalizes the Calabi-Eckmann manifolds. These examples will be needed in the next section in order to realize complex types of cubic forms as cup-forms of complex 3-folds.

4.1 CHERN NUMBERS OF ALMOST COMPLEX STRUCTURES

Let X be a closed, oriented, 6-dimensional differentiable manifold. The tangent bundle of X is induced by a classifying map $t_X : X \to BSO(6)$ which is unique up to homotopy. By an almost complex structure on X we mean the homotopy class $[\tilde{t}_X]$ of a lifting $\tilde{t}_X : X \to BU(3)$ of t_X to $BU(3)$.