3.2 The GIT quotient $S^3 H_C^v//SL(H_C)$

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REMARK 6. It is not difficult to show that F^t is not injective if and only if there exists a proper quotient $\bar{H}_{\mathbf{C}}$ of $H_{\mathbf{C}}$, and a form $\bar{F} \in S^3\bar{H}_{\mathbf{C}}^{\vee}$ whose pull-back to $H_{\mathbf{C}}$ is the given form F. This means that the Hessians of cubic polynomials $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ which 'do not depend on all variables' are automatically zero.

The converse holds for forms in $b \leqslant 4$ variables, but not in general [G/N].

3.2 THE GIT QUOTIENT $S^3H_{\mathbf{C}}^{\vee}//_{SL(H_{\mathbf{C}})}$

Let $V := S^3H_{\mathbf{C}}^{\vee}$ be the vector space of complex cubic forms. The reductive group $G := SL(H_{\mathbf{C}})$ acts rationally on V, and therefore has a finitely generated ring $\mathbf{C}[V]^G$ of invariants [H]. The inclusion $\mathbf{C}[V]^G \subset \mathbf{C}[V]$ induces a regular map $\pi : V \to V//_G$ onto the affine variety $V//_G$ with coordinate ring $\mathbf{C}[V]^G$. It is well known that π is a categorical quotient, which is G-closed and G-separating, so that $V//_G$ parametrizes precisely the closed G-orbits in V. Recall that a point $v \in V$ is semi-stable if $o \notin \overline{G \cdot v}$, and that v is stable if $G \cdot v$ is closed in V and the isotropy group G_v is finite [M/F]. Denote the G-invariant, open subsets of semistable (stable) points in V by $V^{ss}(V^s)$.

The complement $V \setminus V^{ss} = \pi^{-1}(\pi(0))$ consists of 'Nullformen', i.e. forms for which all polynomial invariants vanish. The open subset of stable points, which includes in particular all non-singular forms, has a geometric quotient, given by the restricted map $\pi \mid V^s : V^s \to \pi(V^s)$.

REMARK 7. Let $A_o \in GL(H)$ be a fixed automorphism of determinant $\det A_o = -1$, e.g. $A_o = -id_H$ if b is odd. A_o induces a $\mathbf{Z}_{/2}$-action on $S^3H^{\vee}/_{SL(H)}$ and on $S^3H_{\mathbf{C}}^{\vee}/_{SL(H_{\mathbf{C}})}$, for which the map c is equivariant.

Let $\hat{G} \subset GL(H_{\mathbf{C}})$ be the semi-direct product of $SL(H_{\mathbf{C}})$ and $\mathbf{Z}_{/2}$ generated by A_o and $SL(H_{\mathbf{C}})$. The invariant ring $\mathbf{C}[V]^{\hat{G}}$ has an important topological interpretation: it consists of all polynomial invariants of complex cohomology rings of 1-connected, closed, oriented 6-dimensional manifolds with torsion-free homology.

EXAMPLE 5. Binary cubics ($b = 2$)

Choose linear coordinates X, Y on $H_{\mathbf{C}}$, and write a cubic polynomial $f \in \mathbf{C}[X, Y]_3$ in the form $f = a_0X^3 + 3a_1X^2Y + 3a_2XY^2 + a_3Y^3$.

We use a_0, a_1, a_2, a_3 as coordinates on $S^3H_{\mathbf{C}}^{\vee}$, so that $\mathbf{C}[S^3H_{\mathbf{C}}^{\vee}] = \mathbf{C}[a_0, a_1, a_2, a_3]$. The discriminant $\Delta(f)$ of f is a homogeneous polynomial of degree 4 in the coefficients a_0, a_1, a_2, a_4, explicitly given by $\Delta(f) = a_0^2 a_3^2 - 3a_1^2 a_2^2 - 6a_0 a_1 a_2 a_3 + 4a_0 a_2^3 + 4a_1^3 a_3$.

The discriminant generates the ring of $SL(H_{\mathbf{C}})$-invariants,

$$\mathbf{C}[S^3 H_{\mathbf{C}}^{\vee}]^{SL(H_{\mathbf{C}})} = \mathbf{C}[\Delta] ,$$

and it is easy to see that Δ is also $\mathbf{Z}_{/2}$-invariant. A cubic form f is stable if and only if it is semistable, if and only if it is non-singular [N]. The cone of nullforms $\pi^{-1}(\pi(0))$ is the affine hypersurface $(\Delta)_0 \subset S^3 H_{\mathbf{C}}^{\vee}$; it has a nice geometric interpretation in terms of the Hessian. The Hessian of the cubic f is the quadratic form

$$H_f = 6^2[(a_0 a_2 - a_1^2)X^2 + (a_0 a_3 - a_1 a_2)XY + (a_1 a_3 - a_2^2)Y^2] .$$

The set of forms f with vanishing Hessians H_f form the affine cone over the rational normal curve in $\mathbf{P}(S^3 H_{\mathbf{C}}^{\vee})$; the hypersurface of nullforms is the cone over the tangential scroll of this curve. There are 4 different types of $SL(H_{\mathbf{C}})$-orbits in $S^3 H_{\mathbf{C}}^{\vee}$, represented by the normal forms $XY(X + \lambda Y)$, $X^2 Y, X^3, 0$. The first type is stable, the others are nullforms, the orbits of X^3 and 0 have vanishing Hessians.

EXAMPLE 6. Ternary cubics ($b = 3$)

The ring of $SL(H_{\mathbf{C}})$-invariants of ternary cubics is a weighted polynomial ring in 2 variables, $\mathbf{C}[S^3 H_{\mathbf{C}}^{\vee}]^{SL(H_{\mathbf{C}})} = \mathbf{C}[S, T]$ whose generators S, T have been found by S. Aronhold [A]. S is a homogeneous polynomial of degree 4 in the coefficients of a cubic f, T is homogeneous of degree 6, both polynomials are $\mathbf{Z}_{/2}$-invariant. For a cubic of the form $f = aX^3 + bY^3 + cZ^3 + 6dXYZ$, S and T are given by $S = 4d(d^3 - abc)$ and $T = 8d^6 + 20abc(d^3 - abc)$ respectively [P]. The general formulae, which take two pages to write down, can be found in the book of Sturmfels [St]. The discriminant of a form f is homogeneous of degree 12 in the coefficients of f; in terms of Aronhold's invariants S, T it is simply given by $\Delta = S^3 - T^2$. We obtain the following overall picture: The GIT quotient for ternary cubics is an affine plane $\mathbf{A}^2$ with coordinates S, T. The complement $\mathbf{A}^2 \setminus (\Delta)_0$ of the discriminant curve is the geometric quotient of stable cubics. The π-fibers over a point $(S, T) \neq (0, 0)$ on the discriminant curve $(\Delta)_0$ consist of 3 types of $SL(H_{\mathbf{C}})$-orbits: nodal cubics with normal form $X^3 + Y^3 + 6\alpha XYZ$, reducible cubics formed by a smooth conic and a transversal line (normal form: $X^3 + 6\alpha XYZ$), and cubics consisting of three lines in general position (normal form: $6\alpha XYZ$); these cubics are properly semi-stable for $\alpha \neq 0$ with Aronhold invariants $S = 4\alpha^4$, $T = 8\alpha^6$. The fiber of π over 0 contains 6 orbits with normal forms

$$Y^2Z - X^3, Y(X^2 - YZ), XY(X+Y), X^2Y, X^3,$$

and 0, of which the last 4 types have vanishing Hessians. For more details we refer to H. Kraft's book [Kr].

REMARK 8. The natural $\mathbf{C}^*$-action $f \to \lambda \cdot f$ on cubic forms induces a weighted action on the GIT quotient $S^3H_{\mathbf{C}}^\vee/_{SL(H_{\mathbf{C}})}$, $\lambda \cdot (S, T) = (\lambda^4 S, \lambda^6 T)$. The associated weighted projective space $\mathbf{P}^1(4, 6)$ with homogeneous coordinates $<S, T>$ is the good quotient for semi-stable plane cubic curves. Its affine part $\mathbf{P}^1\backslash(\Delta)_0$ is the moduli space of genus-1 curves. The $PGL(H_{\mathbf{C}})$-invariant $J := \frac{S^3}{\Delta}$ gives the J-invariant of the corresponding curve.

3.3 ARITHMETICAL ASPECTS

Let $c: S^3H^\vee/_{SL(H)} \to S^3H_{\mathbf{C}}^\vee/_{SL(H)}$ be the map which associates to the $SL(H)$-orbit $<F>$ of a symmetric trilinear form $F \in S^3H^\vee$ the $SL(H_{\mathbf{C}})$-orbit $<F>_{\mathbf{C}}$ of its complexification. The c-fiber over $<F>_{\mathbf{C}}$ can be identified with the subset $(SL(H_{\mathbf{C}}) \cdot F \cap S^3H^\vee)/_{SL(H)}$ of $S^3H^\vee/_{SL(H)}$. C. Jordan has shown that these subsets are finite provided the cubic form $f \in \mathbf{C}[H_{\mathbf{C}}]_3$ associated to F has a non-vanishing discriminant [J1]. Jordan's original proof, which is only two pages long, is somewhat hard to follow. The following theorem of A. Borel and Harish-Chandra provides, however, a vast generalization of Jordan's finiteness result:

THEOREM 3 (Borel/Harish-Chandra). *Let G be a reductive $\mathbf{Q}$-group, $\Gamma \subset G$ an arithmetic subgroup, $\xi: G \to GL(V)$ a $\mathbf{Q}$-morphism, and $L \subset V$ a Γ-invariant sublattice of $V_{\mathbf{Q}}$. If $v \in V$ has a closed G-orbit in V, then $G_v \cap L/_\Gamma$ is a finite set.*

Proof. [B].

COROLLARY 4. *Let $F \in S^3H^\vee$ be a symmetric trilinear form on H. If the $SL(H_{\mathbf{C}})$-orbit of F in $S^3H_{\mathbf{C}}^\vee$ is closed, then the fiber $c^{-1}(<F>_{\mathbf{C}})$ over $<F>_{\mathbf{C}}$ is finite.*

To check whether a $SL(H_{\mathbf{C}})$-orbit $SL(H_{\mathbf{C}}) \cdot F$ is closed in $S^3H_{\mathbf{C}}^\vee$, one has a generalization of the Hilbert-criterion [Kr]: $SL(H_{\mathbf{C}}) \cdot F$ is closed in $S^3H_{\mathbf{C}}^\vee$ if and only if for every 1-parameter subgroup $\lambda: \mathbf{C}^* \to SL(H_{\mathbf{C}})$, for