4. Invariants of complex 3-folds

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The number of integral classes in these orbits is therefore finite. We have, however, an even stronger finiteness theorem for stable ternary cubics:

PROPOSITION 7. *Let H be a free $\mathbf{Z}$-module of rank 3. There exist only finitely many classes of symmetric trilinear forms $F \in S^3 H^\vee$ with a fixed discriminant $\Delta \neq 0$.*

Proof. In terms of Arnhold's invariants S and T, Δ is given by $\Delta = S^3 - T^2$. By a theorem of C. Siegel [Si], the diophantine equation $S^3 - T^2 = \Delta$ has only finitely many integral solution (S, T) for any integer $\Delta \neq 0$. For each of these solutions the corresponding point in $S^3 H_{\mathbf{C}}^\vee / _{SL(H_{\mathbf{C}})}$ lies outside of the discriminant curve, so that the π-fiber over it is a closed $SL(H_{\mathbf{C}})$-orbit. The finiteness of the class number then follows from the Borel/Harish-Chandra theorem.

A famous special case of Siegel's theorem is Bachet's equation $S^3 - T^2 = 2$; it has only the two obvious solutions $(3, \pm 5)$.

REMARK 10. To get finiteness results for ternary cubic forms it is not sufficient to fix the J-invariant (instead of the discriminant): The forms $f_m = X^3 + XZ^2 + Z^3 + mY^2Z$, $m \in \mathbf{Z}\backslash\{0\}$, all have the same J-invariant, but they are not equivalent, even over $\mathbf{Q}$, since they have bad reduction at different primes $p \mid m$.

4. INVARIANTS OF COMPLEX 3-FOLDS

In this section we begin to investigate the topology of 1-connected, compact, complex 3-folds. After a brief discussion of the possible systems of Chern numbers of almost complex 6-manifolds, we study the behaviour of the topological invariants of complex 3-folds under certain standard constructions, like e.g. branched coverings, or blow-ups of points and curves. Then we describe some interesting examples of 1-connected, non-Kählerian 3-folds, including a new construction method which generalizes the Calabi-Eckmann manifolds. These examples will be needed in the next section in order to realize complex types of cubic forms as cup-forms of complex 3-folds.

4.1 CHERN NUMBERS OF ALMOST COMPLEX STRUCTURES

Let X be a closed, oriented, 6-dimensional differentiable manifold. The tangent bundle of X is induced by a classifying map $t_X : X \to BSO(6)$ which is unique up to homotopy. By an almost complex structure on X we mean the homotopy class $[\tilde{t}_X]$ of a lifting $\tilde{t}_X : X \to BU(3)$ of t_X to $BU(3)$.

PROPOSITION 8. *Every closed, oriented, 6-dimensional C^∞-manifold X without 2-torsion in $H^3(X, \mathbf{Z})$ admits an almost complex structure. There is a 1-1 correspondence between almost complex structures on X and integral lifts $W \in H^2(X, \mathbf{Z})$ of $w_2(X)$. The Chern classes c_i of the almost complex manifold (X, W) are given by $c_1 = W$, $c_2 = \frac{1}{2}(W^2 - p_1(X))$.*

Proof (cf. [W]). The obstructions against lifting t_X to $BU(3)$ lie in the cohomology groups $H^{i+1}(X, \pi_i(SO(6)/_{U(3)}), i = 0, 1, \ldots, 5$. Since $SO(6)/_{U(3)} = \mathbf{P}^3$ has only one nontrivial homotopy group $\pi_2(SO(6)/_{U(3)}) \cong \mathbf{Z}$ in dimensions $i \leqslant 5$, there is in fact only one obstruction $o(t_X) \in H^3(X, \mathbf{Z})$, and this obstruction can be identified with the image of $w_2(X)$ under the Bockstein homomorphism $\beta : H^2(X, \mathbf{Z}_{/2}) \to H^3(X, \mathbf{Z})$. Since $H^3(X, \mathbf{Z})$ has no 2-torsion by assumption, $\beta w_2(X)$ must be equal to zero, so that X has at least one almost complex structure $[\tilde{t}_X] \in [X, BU(3)]$. Standard homotopy arguments show now that the map, which assigns to an almost complex structure $[\tilde{t}_X]$ its first Chern class $\tilde{t}_X^* c_1$, induces a 1-1 correspondence between integral lifts $W \in H^2(X, \mathbf{Z})$ of $w_2(X)$ and homotopy classes of liftings of $[t_X]$ to $BU(3)$.

The second Chern class c_2 of the almost complex manifold (X, W) is determined by $W^2 - 2c_2 = p_1(X)$.

The Chern numbers $c_1^3, c_1 c_2, c_3$ of an almost complex manifold X of real dimension 6 satisfy the following congruences: $c_1^3 \equiv 0 (\mathrm{mod}\, 2)$, $c_1 c_2 \equiv 0 (\mathrm{mod}\, 24)$, $c_3 \equiv 0 (\mathrm{mod}\, 2)$. Conversely, given a triple (a, b, c) of integers $a \equiv 0 (\mathrm{mod}\, 2)$, $b \equiv 0 (\mathrm{mod}\, 24)$, and $c \equiv 0 (\mathrm{mod}\, 2)$, there always exists an almost complex manifold X of dimension 6 with Chern numbers $c_1^3 = a, c_1 c_2 = b, c_3 = c$.

It is not totally clear, however, that one can find a *connected* manifold X with prescribed Chern numbers [H1].

PROPOSITION 9. *Every triple $(a, b, c) \in \mathbf{Z}^{\oplus 3}$ satisfying $a \equiv 0 (\mathrm{mod}\, 2)$, $b \equiv 0 (\mathrm{mod}\, 24)$, $c \equiv 0 (\mathrm{mod}\, 2)$ is realizable as the Chern numbers of an almost complex 6-manifold.*

Proof. Consider the complete intersection $V(f, g) \subset \mathbf{P}^5$ defined by the polynomials $f(z) = z_0^2 + z_1^2 + 2z_2^2 - z_3^2 - z_4^2 - 2z_5^2$, and $g(z) = z_0^4 + z_1^4 + 2z_2^4 - z_3^4 - z_4^4 - 2z_5^4$ [We]. $V(f, g)$ is a singular 3-fold with 90 ordinary double points, and every small resolution V of these nodes is a (not necessarily projective) Calabi-Yau 3-fold with Euler number 4. Suppose now that a prescribed triple $(a, b, c) \in \mathbf{Z}^{\oplus 3}$ is realized by a possibly disconnected almost complex manifold $X = \coprod_{i \in I} X_i$. If we form the connected sum

X' of the X_i, we obtain a connected almost complex manifold X' with Chern numbers $c_1^3 = a, c_1c_2 = b$, but with $c_3 = c - 2(|I| - 1)$.

If $|I| > 1$ take the connected sum of X' with $|I| - 1$ copies of the complex manifold V. Since V is Calabi-Yau, the Chern numbers c_1^3 and c_1c_2 remain unchanged, whereas the Euler number of $X' \#_{|I|-1} V$ becomes $c_3 = c$.

REMARK 11. The above argument has been suggested by F. Hirzebruch after talk at the MPI, in which one of us had sketched a less geometric proof of the proposition.

There is another question which is related to the result above: Fix a closed, oriented, 6-dimensional differentiable manifold X. Which pairs (a, b) of integers with $a \equiv 0 (\mathrm{mod}\, 2)$ and $b \equiv 0 (\mathrm{mod}\, 24)$ occur as Chern numbers c_1^3 and c_1c_2 of almost complex structures on X, and in how many ways?

For manifolds with $b_2(X) = 1$ the Chern numbers determine the almost complex structure. For manifolds with $b_2 > 1$ this is no longer true. It is possible to construct infinitely many distinct almost complex structures with the same Chern numbers on a hypersurface of bidegree (3, 3) in $\mathbf{P}^2 \times \mathbf{P}^2$.

An almost complex structure $[\tilde{t}_X]$ on a differentiable 6-manifold X is said to be integrable if $\tilde{t}_X$ is homotopic to the classifying map of a complex 3-fold. We are not aware of any example of an almost complex 6-manifold which is known not be integrable. On the other hand, it is also unknown whether or not the Chern numbers c_1^3, c_1c_2 of integrable almost complex manifold are topological invariants. The following remark might therefore be of some interest:

PROPOSITION 10. *If the Chern numbers of complex 3-folds are topological invariants, then there exist almost complex structures which are not integrable.*

Proof. Consider a closed, oriented differentiable 6-manifold X without 2-torsion in $H^3(X, \mathbf{Z})$. Fix any almost complex structure on X with first Chern class $W \in H^2(X, \mathbf{Z})$.

Every element $x \in H^2(X, \mathbf{Z})$ defines a new almost complex structure on X with first Chern class $W + 2x$, and it is easy to see that these two almost complex structures have the same Chern numbers if and only if x satisfies the equations $p_1(X) \cdot x = 0$, and $3W^2 \cdot x + 6W \cdot x^2 + 4x^3 = 0$.

Suppose now (X, W) is integrable, $p_1(X) \neq 0$, and choose $x \in H^2(X, \mathbf{Z})$ such that $p_1(X) \cdot x \neq 0$. Then clearly, either none of the almost complex manifolds $(X, W + 2x)$ is integrable, or the Chern numbers of complex 3-folds are not topologically invariant.

REMARK 12. It is very likely that there exist non-integrable almost complex structures on manifolds X as above, but probably this is hard to prove. It is also not unlikely that the Chern numbers of complex 3-folds are not topological invariants. A possible way to check this would be, to run a computer search for 3-folds given by certain standard constructions.

4.2 STANDARD CONSTRUCTIONS

For later use we investigate the topological invariants of complex 3-folds which can be obtained by certain simple standard constructions like complete intersections, simple cyclic coverings, blow-ups of points and curves, and projective bundles.

PROPOSITION 11 (Libgober/Wood). *Let* $X \subset \mathbf{P}^{3+r}$ *be a smooth complete intersection of multidegree* $\underline{d} = (d_1, ..., d_r)$. *Choose a normalized basis* $e \in H^2(X, \mathbf{Z})$, *and let* $\varepsilon \in H^4(X, \mathbf{Z})$ *be defined by* $\varepsilon(e) = 1$. *Then the invariants of* X *are:*

$F_X(xe) = dx^3$ *where* $d = \prod_{i=1}^r d_i$, $w_2(X) \equiv (4 + r - \sum_{i=1}^r d_i)e$,

$p_1(X) = d(4 + r - \sum_{i=1}^r d_i^2)\varepsilon$, *and*

$$b_3(X) = 4 - \tfrac{d}{6}[(4 + r - \textstyle\sum_{i=1}^r d_i)^3 - 3(4 + r - \sum_{i=1}^r d_i)(4 + r - \sum_{i=1}^r d_i^2) + 2(4 + r - \sum_{i=1}^r d_i^3)] .$$

Proof. [L/W].

PROPOSITION 12. *Let* X *be a smooth, 1-connected, complex projective 3-fold, and let* $\pi : X' \to X$ *be a simple cyclic covering of degree* d *branched along a non-singular ample divisor* $B \in |L^{\otimes d}|$. X' *is smooth, projective, 1-connected, and* $\pi^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ *is an isomorphism. The invariants of* X *and* X' *are related by the formulae:*

$$(\pi^*)^* F_{X'} = dF_X, \; w_2(X') - \pi^* w_2(X) \equiv (d-1)\pi^* c_1(L),$$

$$p_1(X') - \pi^* p_1(X) = (1-d)(1+d)\pi^* c_1(L)^2, \; \textit{and}$$

$$b_3(X') = db_3(X) + (d-1)(b_2(B) - 2b_2(X)) .$$

Proof. X' is clearly smooth and projective. By a theorem of M. Cornalba $\pi : X' \to X$ is a 3-equivalence, i.e. $\pi_* : \pi_i(X') \to \pi_i(X)$ is bijective for $i \leqslant 2$, and surjective for $i = 3$ [*Co*]. X' is therefore 1-connected, and $\pi^* : H^2(X, \mathbf{Z}) \to H^2(X', \mathbf{Z})$ is an isomorphism. The relation between $F_{X'}$ and F_X is obvious, whereas the formula for $b_3(X')$ follows from $\pi_1(B) = \{1\}$ and standard properties of Euler numbers.

In order to calculate $w_2(X')$ and $p_1(X')$ we compute the Chern classes of $X': c_1(X') - \pi^* c_1(X) = (1-d)\pi^* c_1(L), c_2(X') - \pi^* c_2(X) = (1-d)\pi^*[c_1(X)c_1(L) - dc_1(L)^2]$.

The latter formulae follow from the description of X' as a divisor in the total space of the line bundle L.

EXAMPLE 9. Let X be a d-fold, simple cyclic covering of $\mathbf{P}^3$ branched along a smooth surface $B \subset \mathbf{P}^3$ of degree $dl, l \geqslant 1$. Let $e \in H^2(X, \mathbf{Z})$ correspond to the preimage of a plane in $\mathbf{P}^3$. The invariants of X are then given by:

$$F_X(xe) = dx^3,\ w_2(X) \equiv (4 + (1-d)l)e, p_1(X) = d[4 + (1-d)(1+d)l^2]\varepsilon \quad (\varepsilon(e) = 1), b_3(X) = (d-1)(d^2l^2 - 4dl + 6)dl\ .$$

PROPOSITION 13. *Let* $\sigma: \hat{X} \to X$ *be the blow-up of a complex 3-fold* X *in a point, and let* $e \in H^2(\hat{X}, \mathbf{Z})$ *be the class of the exceptional divisor. The invariants of* $\hat{X}$ *and* X *are related by the following formulae:*

$$F_{\hat{X}}(\sigma^* h + xe) = F_X(h) + x^3\ \forall\ h \in H^2(X, \mathbf{Z}), x \in \mathbf{Z},\ w_2(\hat{X}) = \sigma^* w_2(X),$$
$$p_1(\hat{X}) = \sigma^* p_1(X) + 4(e^2 - \sigma^* c_1(X) \cdot e),\ b_3(\hat{X}) = b_3(X)\ .$$

Proof. Standard arguments, see [G/H]. The Chern classes are related by $c_1(\hat{X}) = \sigma^* c_1(X) - 2e, c_2(\hat{X}) = \sigma^* c_2(X)$.

PROPOSITION 14. *Let* $\sigma: \hat{X} \to X$ *be the blow-up of a complex 3-fold* X *along a smooth curve* C *of genus* g, *and let* $e \in H^2(\hat{X}, \mathbf{Z})$ *be the class of the exceptional divisor. The invariants of* $\hat{X}$ *and* X *are related by:*

$$F_{\hat{X}}(\sigma^* h + xe) = F_X(h) - 3h \cdot Cx^2 - deg N_{C/X} x^3\ \forall\ h \in H^2(X, \mathbf{Z}),$$
$$x \in \mathbf{Z},\ w_2(\hat{X}) \equiv \sigma^* w_2(X) + e, p_1(\hat{X}) = \sigma^* p_1(X) + (e^2 - 2\sigma^* C),$$
$$b_3(\hat{X}) = b_3(X) + 2g\ .$$

Proof. [*G/H*]. The Chern classes are given by $c_1(\hat{X}) = \sigma^* c_1(X) - c, c_2(\hat{X}) = \sigma^*(c_2(X) + C) - \sigma^* c_1(X) \cdot e$.

PROPOSITION 15. *Let* E *be a holomorphic vector bundle of rank 2 with Chern classes* $c_i(E), i = 1, 2$ *over a 1-connected, compact complex surface* Y, *and let* $\pi: \mathbf{P}(E) \to Y$ *be the projective bundle of lines in the fibers of* E. *The cup-form of* $\mathbf{P}(E)$ *is given by*

$$F_{\mathbf{P}(E)}(h + x\xi) = x[(3h^2) - (3c_1(E) \cdot h)x + (c_1(E)^2 - c_2(E))x^2]\ ,$$

where $\xi = c_1(\mathcal{O}_{\mathbf{P}(E)}(1))$, $h \in H^2(Y, \mathbf{Z})$, *and* $x \in \mathbf{Z}$. *The other topological invariants of* $\mathbf{P}(E)$ *are:*

$$w_2(\mathbf{P}(E)) \equiv \pi^*(w_2(Y) + c_1(E)), p_1(E))$$
$$= \pi^*[c_1(Y)^2 - 2c_2(Y) + c_1(E)^2 - 4c_2(E)],\ b_3(\mathbf{P}(E)) = 0\ .$$

Proof. The Leray-Hirsch theorem identifies the cohomology ring $H^*(\mathbf{P}(E), \mathbf{Z})$ with the ring $H^*(Y, \mathbf{Z})[\xi]/_{<\xi^2 + c_1(E)\cdot\xi + c_2(E)>}$; this determines the cup-form. In order to calculate the characteristic classes one uses the exact sequence $0 \to \mathcal{O}_{\mathbf{P}(E)} \to \pi^*E \otimes \mathcal{O}_{\mathbf{P}(E)}(1) \to T_{\mathbf{P}(E)} \to \pi^*T_Y \to 0$. $b_3(\mathbf{P}(E)) = 0$ follows from $b_1(Y) = 0$ and the Leray-Hirsch theorem.

4.3 Examples of 1-connected non-Kählerian 3-folds

Recall that the Hessian of a symmetric trilinear form $F \in S^3H^\vee$ on a free $\mathbf{Z}$-module H of finite rank was defined as the composition $H_F: H \overset{F^t}{\to} S^2H^\vee \overset{\text{disc}}{\to} \mathbf{Z}$. In terms of coordinates $\xi_1, \ldots, \xi_b$ on H it is given by the determinant $\det\left(\frac{\partial^2 f}{\partial\xi_i \partial\xi_j}\right)$, where $f \in \mathbf{C}[H_\mathbf{C}]_3$ is the homogeneous cubic polynomial associated with F.

Proposition 16. *Let* F *be a symmetric trilinear form whose Hessian vanishes identically. Then* F *is not realizable as cup-form of a Kählerian 3-fold.*

Proof. Let X be a complex 3-fold with a Kähler metric g. The Kähler class $[\omega_g] \in H^2(X, \mathbf{R})$ defines a multiplication map $\cdot[\omega_g]: H^2(X, \mathbf{R}) \to H^4(X, \mathbf{R})$, which is an isomorphism by the Hard Lefschetz Theorem [G/H]. In section 3.1 we have seen that this is not possible if the Hessian of the cup-form vanishes.

Corollary 6. *Cubic forms* $f \in \mathbf{C}[H_\mathbf{C}]_3$ *which depend on strictly less than* $b = rk_\mathbf{Z} H$ *variables are not realizable as cup-forms of Kählerian 3-folds with* $b_2 = b$.

By considering the Hessian of a cup-form over the reals one obtains further conditions.

Definition 4. *Let* $F \in S^3H^\vee$ *be a symmetric trilinear form on a free* $\mathbf{Z}$*-module of rank* b.

The Hesse cone of F *is the subset* $\mathcal{H}_F \subset H_\mathbf{R}$ *defined by* $\mathcal{H}_F := \{h \in H_\mathbf{R} \mid (-1)^b \det(F^t(h)) < 0\}$.

The index cone $\mathscr{I}_F$ *of* F *is the subset* $\mathscr{I}_F := \{h \in \mathscr{H}_F \mid F^t(h) \in S^2 H^\vee_{\mathbf{R}}$ *has signature* $(1, -1, \ldots, -1)\}$.

Clearly $\mathscr{I}_F$ is an open subcone of $\mathscr{H}_F$ which coincides with $\mathscr{H}_F$ iff $b \leqslant 2$.

THEOREM 5. *Let* $F_X \in S^3 H^2(X, \mathbf{Z})^\vee$ *be the cup-form of a smooth projective 3-fold with* $h^{0,2}(X) = 0$. *Then* F_X *has a non-empty index cone.*

Proof. Let $h \in H^2(X, \mathbf{Z})$ be the dual class of a hyperplane section Y in some projective embedding. The inclusion $i: Y \hookrightarrow X$ induces a monomorphism $i^*: H^2(X, \mathbf{Z}) \to H^2(Y, \mathbf{Z})$ by the weak Lefschetz theorem. The symmetric bilinear form $F^t_X(h) \in S^2 H^2(X, \mathbf{Z})^\vee$ is simply the pull-back of the cup-form of Y under the inclusion i^*; it is therefore non-degenerate by the Hard Lefschetz theorem [L]. Applying the Hodge index theorem to Y we see that the real bilinear form $F^t_X(h) \in S^2 H^2(X, \mathbf{R})^\vee$ must have one positive and $b - 1$ negative eigenvalues. In other words: $h \in I_{F_X}$.

REMARK 13. This result has two applications: it provides topological 'upper bounds' for the ample cone of a projective 3-fold with $h^{0,2} = 0$, and if gives further restrictions on symmetric trilinear forms to be realizable as cup-forms of projective 3-folds with $h^{0,2} = 0$ if $b \geqslant 4$.

These applications will be discussed in section 5.

We will now describe examples of 1-connected, non-Kählerian, complex 3-folds and determine their topological structure.

EXAMPLE 10 (Calabi-Eckmann). E. Calabi and B. Eckmann have defined complex structures X_τ, depending on a parameter τ, on the product $S^3 \times S^3$ [C/E]. Their manifolds are principal fiber bundles over $\mathbf{P}^1 \times \mathbf{P}^1$ whose fiber and structure group is the elliptic curve $E_\tau = \mathbf{C}/_{\mathbf{Z} \oplus \mathbf{Z}\tau}$, $\mathrm{Im}(\tau) > 0$.

The Calabi-Eckmann manifolds are homogeneous, non-Kählerian 3-folds of algebraic dimension 2.

EXAMPLE 11 (Maeda). H. Maeda has generalized the Calabi-Eckmann construction. He constructed fiber bundles X'_τ over Hirzebruch surfaces $\mathbf{F}_n$, $n \geqslant 0$, whose fiber and structure group are an elliptic curve E_τ and $\mathrm{Aut}(E_\tau)$ respectively [M]. X'_τ is again diffeomorphic to $S^3 \times S^3$, and therefore non-Kählerian. Maeda's manifolds X'_τ are homogeneous if and only if $n = 0$ in which case they are Calabi-Eckmann 3-folds.

The Calabi-Eckmann construction can also be generalized in the following way:

Let $S^2 \tilde{\times} S^4$ be the non-trivial S^4-bundle over S^2, i.e. $S^2 \tilde{\times} S^4$ is the unique 1-connected, closed, oriented, differentiable 6-manifold with $H_2(S^2 \tilde{\times} S^4, \mathbf{Z}) \cong \mathbf{Z}$ and $b_3 = 0$, whose cup-form and Pontrjagin class vanish, but whose Stiefel-Whitney class w_2 is non-zero.

THEOREM 6. *For any integer* $b \geqslant 0$ *there exist compact complex 3-folds* X_b, *and* $X_b^{\sim}$ *if* $b \geqslant 1$, *which are homeomorphic to* $\#_b S^2 \times S^4 \#_{b+1} S^3 \times S^3$, *and* $S^2 \tilde{\times} S^4 \#_{b-1} S^2 \times S^4 \#_{b+1} S^3 \times S^3$.

Proof. Let Y be a 1-connected, compact complex surface with $p_g(Y) = 0$ and $b_2(Y) \geqslant 2$, and let $E = \mathbf{C}/_\Gamma$ be the elliptic curve associated to the lattice $\Gamma \subset \mathbf{C}$. We want to construct the required 3-folds as total spaces of principal E-bundles over Y. Let $c: H_2(Y, \mathbf{Z}) \to \Gamma$ be an arbitrary epimorphism. The corresponding cohomology class $c \in H^2(Y, \Gamma)$ defines a topological principal bundle over Y with fiber and structure group $E = \mathbf{C}/_\Gamma$ as follows immediately from the identification of the classifying space $BE \simeq K(\Gamma, 2)$.

Let $\mathcal{O}_Y(E)$ be the sheaf of germs of holomorphic maps from Y to E. We have a short exact sequence $0 \to \Gamma \to \mathcal{O}_Y \to \mathcal{O}_Y(E) \to 0$ and a corresponding exact cohomology sequence

$$\to H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y(E)) \xrightarrow{\delta} H^2(Y, \Gamma) \to H^2(Y, \mathcal{O}_Y) \to$$

By our assumptions δ is an isomorphism, so that every topological principal E-bundle admits a holomorphic structure. Let X be the total space of such a bundle corresponding to a surjective map $c: H_2(Y, \mathbf{Z}) \to \Gamma$. The homotopy sequence of the fibration $p: X \to Y$ yields the sequence

$$0 \to \pi_2(X) \xrightarrow{p_*} \pi_2(Y) \to \pi_1(E) \to \pi_1(X) \xrightarrow{p_*} \pi_1(Y) \to 0 .$$

Since Y is 1-connected, $\pi_2(Y)$ can be identified with $H_2(Y, \mathbf{Z})$, and then the boundary map $\pi_2(Y) \to \pi_1(E)$ becomes the characteristic map $c: H_2(Y, \mathbf{Z}) \to \Gamma$ of the bundle. This implies $\pi_1(X) = \{1\}$, whereas $H_2(X, \mathbf{Z})$ is given by: $0 \to H_2(X, \mathbf{Z}) \xrightarrow{p_*} H_2(Y, \mathbf{Z}) \xrightarrow{s} \Gamma \to 0$.

In particular, $H_2(X, \mathbf{Z})$ is free as a submodule of $H_2(Y, \mathbf{Z})$, and by dualizing the last sequence we obtain an identification (via p^*)

$$H^2(X, \mathbf{Z}) = H^2(Y, \mathbf{Z})/_{\Gamma^\vee} .$$

The cup-form F_X of X is therefore trivial. In order to calculate $p_1(X)$ and $w_2(X)$, we use the exact sequence of tangent sheaves: $0 \to T_{X/Y} \to T_X$

$\to p^*T_Y \to 0$. Since $T_{X/Y}$ is a trivial bundle, the characteristic classes of X are simply the pullbacks of the corresponding classes of Y. But the map $p^*: H^4(Y, \mathbf{Z}) \to H^4(X, \mathbf{Z})$ is zero, since $< p^*(\varepsilon) \cup p^*(\alpha), [X] > = < \varepsilon \cup \alpha, p_*[X] > = 0$ for all classes $\varepsilon \in H^4(Y, \mathbf{Z})$, and $\alpha \in H^2(Y, \mathbf{Z})$.

Thus $p_1(X) = 0$, and $w_2(X)$ is the residue of $w_2(Y) \in H^2(Y, \mathbf{Z}/_2)$ modulo $\Gamma^\vee/_{2\Gamma^\vee}$.

The Euler characteristic of X is zero, so that from $b_2(X) = b_2(Y) - 2$ we find $b_3(X) = 2(b_2(Y) - 1)$. The system of invariants associated to the manifold X is therefore given by

$$(b_2(Y) - 1, H^2(Y, \mathbf{Z})/_{\Gamma^\vee}, w_2(Y)\,(\mathrm{mod}\,\Gamma^\vee/_{2\Gamma^\vee}), 0, 0, 0)\ ,$$

i.e. X is diffeomorphic to

$$\#_{b_2(Y)-2} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3 \text{ if } w_2(Y) \in \Gamma^\vee/_{2\Gamma^\vee}\ ,$$

and to $S^2 \tilde{\times} S^4 \#_{b_2(Y)-3} S^2 \times S^4 \#_{b_2(Y)-1} S^3 \times S^3$ if $b_2(Y) \geqslant 3$, and $w_2(Y) \notin \Gamma^\vee/_{2\Gamma^\vee}$.

EXAMPLE 12 (Kato). In the two papers [K1], [K2] M. Kato studies the class of compact, complex 3-folds X containing smooth rational curves with neighborhoods biholomorphic to those of projective lines in $\mathbf{P}^3$. On this class of 3-folds, called class L, he defines a semi-group structure $+$ with neutral element $\mathbf{P}^3$.

Kato's connecting operation $+$ is defined by removing 'lines' $L_i \subset X_i$ from 3-folds $X_i, i = 1, 2$, and by identifying the complements $X_i \backslash L_i$ along open sets $U_i \backslash L_i$ obtained from suitable neighborhoods $U_i \subset X_i$.

Starting with a certain elliptic fiber space X_1 over the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ in a point, he constructs a sequence of 3-folds $X_n := X_1 + X_{n-1}, n \geqslant 2$. The 3-folds X_n are 1-connected spin-manifolds with $H_2(X_n, \mathbf{Z}) = \mathbf{Z}$. Their cup-forms F_{X_n}, and their Pontrjagin classes $p_1(X_n)$ are in terms of a (normalized) generator $e_n \in H^2(X_n, \mathbf{Z})$ and its dual class $\varepsilon_n \in H^4(X_n, \mathbf{Z})$ given by $F_{X_n}(xe_n) = (n-1)x^3$, and $p_1(X_n) = 4(n-1)\varepsilon_n$ $(\varepsilon_n(e_n) = 1)$. The third Betti-number of X_n is $4n$.

In particular, X_1 is diffeomorphic to $S^2 \times S^4 \#_2 S^3 \times S^3$, and X_2 is diffeomorphic to $\mathbf{P}^3 \#_4 S^3 \times S^3$. It is interesting to note that the Chern-numbers c_1^3, c_1c_2 of the X_n are $c_1^3 = 64(1-n)$, $c_1c_2 = 24(1-n)$, i.e. they satisfy $8c_1c_2 = 3c_1^3$. For projective manifolds of general type this equality is characteristic for ball quotients [Y].

EXAMPLE 13 (Twistor spaces). Let $p: Z \to M$ be the twistor fibration of a closed, oriented Riemannian 4-manifold (M, g). Z carries a natural almost complex structure which is integrable if and only if g is self-dual [A/H/S].

Examples of 1-connected 4-manifolds which admit self-dual structures are $S^4, \#_n \mathbf{P}^2$, and $K3$-surfaces.

The total spaces of their twistor fibrations are 1-connected complex 3-folds which may be Moishezon for S^4 and $\#_n \mathbf{P}^2$ [C], but which are usually non-Kähler [Hi]. We leave it to the reader to calculate the topological invariants of these 3-folds. There is an interesting relation between Twistor spaces of connected sums and Kato's connection operation $+$ for class L manifolds [K2], [D/F].

EXAMPLE 14 (Oguiso). In a recent preprint [O1] K. Oguiso constructs examples of 1-connected, Moishezon Calabi-Yau 3-folds with very interesting cup-forms. He proves that for every integer $d \geqslant 1$ there exists a smooth complete intersection X'_d of type $(2, 4)$ in $\mathbf{P}^5$ which contains a non-singular rational curve C_d of degree d with normal bundle $N_{C_d/X_d} = \mathscr{O}_{C_d}(-1)^{\oplus 2}$.

The 3-fold X'_d can now be flopped along C_d, i.e. C_d can be blown up to $\mathbf{P}(N_{C_d/X_d}) \cong \mathbf{P}^1 \times \mathbf{P}^1$, and then 'blown down in the other direction'. The resulting 3-fold X_d is a 1-connected Moishezon manifold with trivial canonical bundle and cup-form F_{X_d} given by $F_{X_d}(xe_d) = (d^3 - 8)x^3$. Here $e_d \in H^2(X_d, \mathbf{Z})$ is the normalized generator corresponding to the strict transform of the negative of a hyperplane section of X'_d. The Pontrjagin class of X_d is $p_1(X_d) = (112 + 4d)\varepsilon_d$ where $\varepsilon_d \in H^4(X_d, \mathbf{Z})$ denotes the generator with $\varepsilon_d(e_d) = 1$. Since the Euler-number does not change under a flop we have $b_3(X_d) = 180$ for every d.

5. COMPLEX 3-FOLDS WITH SMALL b_2

In this section we investigate the following natural problem: Which cubic forms can be realized as cup-forms of compact complex 3-folds? For small b_2 something can be said: Any core of a 1-connected, closed, oriented differentiable 6-manifold with $H_2(X, \mathbf{Z}) \cong \mathbf{Z}$ is homotopy equivalent to the core of a 1-connected complex 3-fold. In the case $b_2 = 2$, at least every discriminant Δ is realizable by a complex manifold. If $b_2 = 3$ we can realize all types of complex cubics with one exception, the union of a smooth conic and a tangent line. In addition to these realization results we prove a finiteness